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Design Research on Fractions

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I. Introduction

Considerable research in Mathematics education in the area of "understanding of fraction" has been conducted in previous research such as those by Streefland(1991), Keijzer(2003), Simon and Tzur (2004). This is because fraction is one of the most difficult mathematical tasks for elementary school students. One reason among many others is that, traditionally, the content of Mathematics in a primary school is mainly based on the product of formal academic Mathematics. Based on the National curriculum, Indonesian elementary school students are supposed to have studied the relations among fractions, such as comparing fractions, equivalence of fractions and simple operations with fractions in terms of addition from grade 3. These fractions' concepts are taught through abstract ways of learning, and therefore, most students are assumed to have sufficient skills to carry out mathematical relations and simple operations in fractions because they have been trained for the skills and should have mastered such procedures without "understanding". This may actually cause a problem for the students in that they can easily forget their initial fractions knowledge whenever they learn fractions at a higher level.

In order to capture this phenomenon, it seems to be necessary to remodel mathematics teaching and learning, especially in fractions. Therefore, we conducted a design research that develops a sequence of fraction activities emphasizing on a shift from "mastering procedures" to "understanding". In proposing this design research, we considered prior research and literature on fractions and also referred to Realistic Mathematics Education (henceforth RME) that was developed in the Netherlands in early 1970s. This method was based on the idea of mathematics as a human activity and as a constructive activity.

Our design research aimed at developing students' understanding on fractions, the relations among fractions such as equivalence of fractions and comparing fractions, as well as simple addition of fractions. In this design research, we introduced three contextual situations and used some models for learning fractions such as the paper bar, string rubber bands and number line models. Underneath, we raised a research question probing the roles of contextual situations and those models which are related to them in learning fractions. However, a sequence of activities based on RME indicates different ways of learning activities. In RME, there are more discussions between teacher and students as well as among students. Then, this approach also leads the teacher to guide the students to construct their ideas in learning and hence raising another research question about the development of teaching and learning fractions.

In order to answer the two research questions, this thesis will be organized in the following ways. First of all, we will initially elaborate on the theoretical framework underlying fractions as our mathematical domain, measurement and RME approach. The elaboration will focus on the design research concerning the relation among fractions as a theme and the use of the RME approach with measurement as a context for the activity, and the paper bar, the string rubber bands and the number line as the models for fractions.

In the next part of this thesis, three phases of the design research such as preparation for the experiment, design experiment and retrospective analysis will be depicted. Also in this part, we will suggest conditions of the students as well as the national curriculum with respect to the research domain.

Subsequently, we develop our Local Instructional Theory (henceforth LIT) and HLT that consist of goals of the research, formulation of the research questions, conjectures about possible fractions learning process, conjectures about possible means of supporting that learning process, potential productive instructional activities, envisioned class room culture and a proactive role by the teacher. The sequence of the activities that was designed in the preparation of the experiment phase was found to undergo changes during the design experiment phase on the basis of daily analysis. Also described in this part will be the pre-test and post-test that we gave to students. The pre-test was given to see whether or not students' prior knowledge was sufficient to involve in the activity. Then the post-test was given to observe the development of students' learning after following the sequence of activities.

After that, we will analyze the result from the design experiment phase featuring affirmation and explanation of the conjecture of the students' learning process. The analysis was triggered to see whether or not our HLT worked. This was also used as a consideration of a new HLT for the following implementation of learning fractions as well.

In the last part of this thesis, we will provide an explanation of roles of the aforementioned models and the students' development in learning fractions. At the end, we will suggest a revisable HLT for better implementation of learning fractions in a classroom setting considering the present research findings.

II. Theoretical Framework

We made a design research on fractions with students in grade 4. As established in the previous chapter, this design research is particularly concerned with the association between fractions as theme and the Realistic Mathematics approach that we used, with measurement as the context of activity and the paper bar, the string rubber bands and the number line as models for fractions. To support this design research, I elaborated some theoretical framework underlying the fractions as our mathematical domain in this chapter.

2.1 Mathematical Phenomenology of fractions

The mathematical phenomenology analyses how a mathematical thought object organizes mathematical phenomena. Based on Freudenthal, mathematical phenomenology is analyzing how a mathematical 'thought object' (a concept, procedure or tool) organizes certain phenomena. There are antitheses in the aspect of phenomenology, such as nooumenon and phainomenon. The mathematical objects are nooumena, but a piece of mathematics can be experienced as a phainomenon. In our case, the fraction itself is the nooumena; however working with fraction or in other words operation with fraction, can be a phainomenon.

"Fractions" are related to breaking: "fracture". Based on that, Freudenthal (1983) defined four aspects of fractions and considered the *"fraction as fracturer"* be one of the aspects. The three other aspects are: *"Fraction as comparer"*, *"Fraction as operator"* and fraction in the most concrete way is as wholes being split into equal parts.

- Fraction as fracturer
  It describes how the magnitudes are divided without a remainder. It consists of comparing quantities and magnitudes by sight or feel, folding and weighing parts in one hand or in a balance. The idea is to involve children in primitive measuring activity before formal measurement is used.
- Fraction as comparer
  Fractions also serve in comparing objects which are separated from each other. Comparing performs certain criteria: directly and indirectly. Directly means that the objects which are to be compared are brought close together. Indirectly means that there is a third object that can be used to mediate between two objects that will be compared

- Fraction as operator

  As described earlier, fraction as fracturer which claims to act on concrete objects that break it into equivalent parts, *the ratio operator*. Fraction can be conceptualized as an operator. ¼ for example can be realized as representing one-fourth of various wholes. Based on Fosnot and Dolk: this concept is important, because its a connection to understanding the double number line. For example, in a chocolate context, the number line consists of the fraction itself and the part of the chocolate.
- Fraction in most concrete way is as wholes being split into equal parts.

  In the most concrete way, fraction themselves is as wholes being split into equal parts through activities such as: splitting, slicing, cutting or coloring.

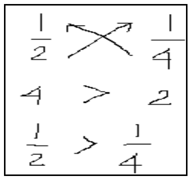
  If we see the symbol of fraction, we can argue that fraction is a positive rational number that symbolized as p/q with $q \neq 0$. Students have conflict in understanding fractions, since they apply the concept of whole number to the arithmetic of fractions. In early learning of rational numbers, students learn fraction as a proportion or a part of a whole.

2.1.1 Equivalence of fractions

Equivalence of fractions is the base of understanding operations with fractions (comparing and addition of fractions). Traditionally, students are taught the equivalence of fractions by multiplying or dividing the fractions by 1 or 2/2, 4/4, etc. For instance, students are asked to find the equivalence of fractions of 2/3. Traditionally, they multiplied it with 1 or maybe 2/2. So, the equivalence of 2/3 is 4/6. By that, students probably understand it algorithmically, but it doesn't make sense for them.

2.1.2 Comparing fractions

If students understand the equivalence of fractions, they are able to compare fractions. There are some cases which the comparing fraction is taught through cross multiplication: the denominator of the first fraction multiplied by the numerator of the second fraction, and the result will be a number for the second fraction to be compared. Then, the denominator of the second fraction is multiplied by the numerator of the first fraction, and the result will be a number for the first fraction to be compared. Below is given an illustration for this explanation. For example, comparing ½ and ¼:



I assume that the big idea of the fastest solution above is using the strategy of multiplying the fraction by 1 or 2/2, 3/3, 4/4, etc so that those two fractions will have the same denominator. Then, the attention is on comparing the numerator of those fractions. Again, it will not make sense and students do not really understand the reason behind that algorithm.

2.1.3 Addition of fractions

After the relation among fractions (comparing fractions and the equivalence of fractions), operation with fractions such as simple addition of fractions is explored. It is considered based on Bezuk and Cramer, 1989, who explain that operations with fractions should be delayed until the concepts and ideas of the order and equivalence of fraction are firmly established.

The most important thing to be considered if we add fractions is the common denominator of those fractions. It would be easier if students had already mastered the idea of the equivalence of fractions after they found the common denominator of those fractions.

In the TAL book, 2007, it said that the comprehensive concept in the “fractions, percentages, decimal and proportions” curriculum is that of proportion. In many cases, the proportions are hidden at an even deeper level, in what Freudenthal refers to as “measuring or proportion number”. We can make a distinction between numbers as labeling numbers, ordering numbers, counting, measuring numbers (in practical situation, we are frequently concerned with question as ”how big” or “how expensive”) and calculation number.

Fractions are almost always proportion numbers. In summary, it is no wonder that Freudenthal concluded that most of the numbers we used in daily life are *measuring and proportion numbers.* Fractions take a central role in this process. Fractions are still believed to have an important place in primary education, based on TAL book. There are two reasons for that:

1. We often figure and think in terms of fractions, even when fractions are not explicitly involved
2. If students understand fractions, they will have a good foundation for proportions, decimal numbers and percentages.

Furthermore, fractions is better introduced first, before decimal and percentages, because if we skip fraction before introducing decimals and percentages, we standardize things from the beginning. If we first introduce decimal and percentages, we shall start right away with tenth and hundredths, instead of introducing hundredths as fraction similar to thirds, quarters, fifth and so on. (TAL Book, 2007).

From the explanation of the TAL Book, we introduced fractions before the decimal and percentages to students' grade 4 (early students who learn fractions). Fractions are almost always proportion numbers. In summary, it is no wonder that Freudenthal concluded that most of the numbers we used in daily life are *measuring and proportion numbers.* Fractions take a central role in this process. Fractions are still believed to have an important place in primary education, based on TAL book. There are two reasons for that:

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From the explanation of the TAL Book, we introduced fractions before the decimal and percentages to students' grade 4 (early students who learn fractions). Besides, considering the mathematical phenomenology on fractions described above, we decided to design a sequence of activities of equivalence, comparing and addition of fractions that keep away from traditional way of teaching. The sequence of activities is emphasized on a shift from mastering procedural and algorithms to "understanding" by given contextual situation.

2.2 Prior Research on Fractions

* Leen Streefland (1991)

He did research on fraction with sharing as the generating activity. His research shown that learner would do better if they start exploring fractions in a more realistic approach, such as a fair-sharing context. The situations had been chosen in such a way that the unit is not the starting point. Leen Streefland in his research used the circle model to represent the pizza. Besides the pizza, another problem that he conducted is sharing a candy bar that was done by dividing one candy bar among three people.

* Ronald Keijzer (2003)

  He used measurement as a context that connected to the number line model. It shows how the teaching of formal fractions is influenced by the use of the number line as a central model for fractions and by the creation of an educational setting in which formal mathematics is discussed in the classroom. He took the number line as the main model for fractions and combined this with whole-class discussions to facilitate the understanding of formal fraction. However, the number line as model hardly fits in with the approach Streefland suggested, because Streefland used the circle as his main model.

* Simon and Tzur (2004)

  Simon and Tzur made the 'equivalence of fraction' teaching sequence containing the following tasks:

1. Draw a rectangle with ½ shaded. Draw lines on the rectangle so that it is divided into sixths. Determine how many sixths are in ½
2. Draw a rectangle with 2/3 shaded. Draw lines on the rectangle so that it is divided into twelves. Determine $\frac{2}{3} = \frac{?}{12}$
3. Draw diagrams to determine : $\frac{3}{4} = \frac{?}{8}$, $\frac{4}{5} = \frac{?}{15}$
4. $\frac{5}{9} = \frac{?}{90}$, $\frac{7}{9} = \frac{?}{72}$, without drawing a diagram, but thinking of drawing of a diagram.
5. $\frac{16}{49} = \frac{?}{147}$, $\frac{13}{36} = \frac{?}{324}$, similar to 4, but now with calculator.
6. Write a calculator protocol for calculating a problem of the form a/b=?/c

The discussion about the prior researches on fractions

The fair sharing as generating activity conducted by Leen Streefland is a good contextual activity for children that have just learned fractions. The model used in the design is the circle model. The circle model is of course nice in showing fraction as part of a whole, but if we go on to the concept of ordering and operation with fraction, this model is not sufficient to guide to those concept. The other model is the model used by Ronald Keijzer, the number line as the central model for fractions within measurement context. There is some evidence that using a bar as a model and a number line as abstraction of the bar can be incorporated into a curriculum that aims at number sense [Keijzer, 1997 in Keijzer, 2003]. However, the number line as a model hardly fits in with the approach Streefland suggested, since Streefland used the circle as the central model. In our opinion the number line model is fit to go on to the operation with fractions concept. The fair sharing and measurement activity can be combined in one activity. For example, by measure a thing to be divided into fair sizes.

Simon and Tzur (2004) created teaching sequences in their design. For tasks 1 to 3, they use drawing a rectangle diagram; and there is a risk if students use their drawing in comparing fraction. For example, if students draw ½ and 2/4 in the different unit rectangle, and if they do not draw carefully, they may think that ½ is larger than 2/4, since based on their drawing the area of ½ is larger than 2/4. Furthermore tasks 4 to 6 were designed to give students the opportunity to develop their abstract understanding directly.

2.3 Measuring

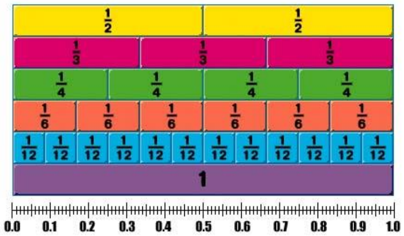
In the TAL book, it said that measuring number can be referred to as proportion number. Why can it be said like that? Since when we are measuring something, we shall see how many times something fits to the thing that we measured. For example, if we measure the length of a classroom using a meter ruler. If the meter ruler fits six times to the length of the classroom then we can say that the length of the classroom is six meters. The six meters show the proportions between the length of the ruler (one meter length) and the length of the class (six meters length). Actually, in practical situation, fractions are almost always proportion numbers.

The fraction activities can be roughly done in a sharing and measuring situation. Between sharing and measuring, there is not really a difference, because the result of sharing can also be regarded as measurement. In this design research, measurement is used as a context to help students get better understanding of the relation of fractions such as comparing fractions, finding the common denominator of fractions and operations with fractions such as addition of fraction. We choose measurement with the string rubber bands and the paper bar model, because those are not far from the number line model, that is a very powerful model to help students do operations with fractions.

2.4 Mathematical Modeling

Models can be divided into two kinds:

1. *The ready to use model* such as fraction stick and wooden fraction circle


(Fractions stick)

2. *The constructed model* such as paper folding, the string rubber bands and the number line.

There is a difference between using those two models. The ready to use model is a model that has been made for the learning process. However, the constructed model is a model that is constructed by students to support the learning process. Learning and working with the ready to use model is not wrong, but there is a risk in using it. If students used the ready to use model, for example the fraction stick, they can just read the symbol of fraction in it (for example 1/12 is smaller than 1/6) and they will lose their reasoning in the relationship between two fractions.

In this design research, we used the constructed model, such as paper folding, the string rubber bands and the number line model. Those models were used to develop the use of landmark fractions for the relation among fractions (exploring comparison of fractions and the equivalence of fractions), exploring the common denominator of fractions and operation with fractions (addition of fractions). Models go through three stages based on Gravemeijer, 1999 and Fosnot and Dolk, 2002 in the Jacob and Fosnot, 2007. Those are:

- Model of the situation
- Model of students' strategies
- Model for thinking and reasoning

In the plan of sequence activities, the paper bar and the string rubber bands are the model of situations of fair dividing the cake and cone cap games contexts. The paper bar is also used as a model for thinking and reasoning of comparing fractions, exploring the common denominator and addition of fractions. The process of making the division leads to reasoning. For instance: by dividing into two parts and then further, student can reach four parts, and ½ can be seen to be 2/4. This model is also helpful to facilitate thinking about the importance of a common denominator for addition of fractions. Another model, which is the number line model, functions as model for thinking and reasoning in exploring the relation among fractions and operation with fractions.

2.5 Social Constructivism and Realistic Mathematics Education

Social constructivism paradigm and Realistic Mathematics Education henceforth referred to as the RME approach, are used in this design. We used this approach because these approaches are rather new in the learning process in Indonesia. Most Indonesian elementary schools use a cognitivism approach and many teachers still use traditional classroom setting. For us, social constructivism is a current reform in mathematics education. Constructivism promotes the principle that mathematics education should exploit invention by the students. However, social constructivism is a learning theory where students are allowed to construct their own understanding of mathematical concept. The role of the teacher is not to lecture or transfer mathematical knowledge, but create situations for students that will encourage them to make the necessary mental constructions. Social constructivism shares similarities with RME. Both of them promote the use of discussion in the classroom between teacher and students, and among students which gives them opportunity to share their ideas and experiences with others. In the sequence activities, we create math congresses in which there will be both teacher-students and student-student discussion and there will be “taken as shared” afterwards. “Taken as shared” is a phrase first used by Prof. Paul Cobb and refers to share meanings which develop through negotiation in the learning environment, and which lead to the development of common; or taken as shared knowledge within a community.

RME is a theory of mathematics education that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing instructional materials for mathematics education [A. Bakker, 2004]. The present form of RME is mostly determined by Freudenthal's view on mathematics [Freudenthal, 1991]. Two of his important points of views are first; mathematics must be connected to reality and second mathematics as human activity. Mathematics must be close to children and be relevant to everyday life situations, so we developed contextual situations that are relevant to and familiar for the students. Besides, the model plays an important role in the learning process. Those points are why this principle called RME (Realistic Mathematics Education). Students should be given the opportunity to experience a process by which mathematics was invented and the teacher role is as a guide who gives guidance to students. Guidance here means striking a delicate balance between the force of teaching and the freedom of learning.

2.6. Didactical Phenomenology

Freudenthal (1973) defines didactical phenomenology as the study of the relation between the phenomena that the mathematical concept represents and the concept itself. In this phenomenology, the focus is on how mathematical interpretations make phenomena accessible for reasoning and calculation. The didactical phenomenology can be viewed as a design heuristic because it suggests ways of identifying possible instructional activities that might support individual activity and whole-class discussions in which the students engage in progressive mathematization [Gravemeijer, 1994]. What didactical phenomenology can do is to prepare the converse approach: starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate this means of organizing. The didactical phenomenology is to be called in to develop plans to realize such an approach.

Didactical phenomenological analysis, therefore, is closely connected with the idea from Freudenthal (1991) in teaching mathematics, where strategy attainment is considered as a process of ‘guided reinvention’: it informs the researcher/designer about a possible reinvention route. He believed that students should be given the opportunity to reinvent the discoveries of our forefathers. Of course we can not expect students in elementary school to do as their forefathers who took centuries to achieve the concept, but under the guidance of the teacher, students can discover for themselves, the concepts that want to be achieved. The guided reinvention is related to constructivist idea. In this design research, the guidance from the teacher is needed in order to achieve the goal of the research. The guidance can be by giving challenging questions to the students, or by orchestrating the discussion in the whole classroom discussion. Guiding here doesn’t means guiding towards a particular answer, because if the teacher does that, it may limit the development of students’ individual ideas. Beside that, a schedule is needed by the teacher in teaching; because it will help organize the lesson. When managing the lesson, the teacher also needs a balance according to student’s ideas and question that she/he will pose.

Social Norms

One of the aspects of the social perspective in learning and teaching is the classroom social norms in which imply describing the classroom participation structure that was established jointly by the teacher and the students. The perspective enables researchers to account for individual learning by viewing it as an act in the classroom interaction. The social norms involved that the teacher need to stimulate students to talk about the math and they do not just accept the answer without justification explanation. Besides, the social norm of the classroom interaction and discussion can be included students' routinely explaining their thinking and reasoning, listening to and questioning others' thinking and responding to others' questions and challenges.

The other students group who listen their friend's solution would make sense of other explanation whether they agreed or disagreed with their friend's group explanation. Likewise, when discussing problem in whole class discussion, students should offer the different solutions from their friend who already contribute in the discussion. These types of norms, however are general classroom norms that may apply to any subject matter [Cobb & Yackel, 1996]

To further clarify, the social norms can be characterized as general norms that are necessary for engaging in classroom discussion.

III. Methodology

3.1 Design Research Methodology

The type of research that we used is design research [Gravemeijer & Cobb, 2001] that is also referred to as developmental research because instructional materials are developed. The centre of design research is a cyclic process of designing instructional sequences, testing and revising them in classroom settings, and then analyzing the learning of the class so that the cycle of design, revision, and implementation can begin again [Gravemeijer & Cobb, 2001]. The purpose of design research in general is to develop theories about both the process of learning and the means that support that learning. The design research cycle consists of 3 phases:

1. The preparation for the experiment
2. The design experiment
3. Retrospective analysis

During the preparation for the experiment phase, we constructed the Hypothetical Learning Trajectory that developed potential sequence activities concerning the goal of the research. This construction was developed based on some supports: we explored and studied prior research on fractions, elaborated mathematical phenomenology related to fractions and discussed these with supervisors and colleagues, and tried out the design with some students in the age range of the experimental subject of the research. The preparation of the design research was done from February 2008 to June 2008. We tried out our design on a grade 3 of Indonesian students (second semester) to see how the planned sequence activities work. There was a short daily retrospective analysis that would result in changes in the Hypothetical Learning Trajectory. Those things were done to give input to us about the probability of the changes and refinements of The Hypothetical Learning Trajectory that we made. In the try out design period, I tried out the design in a small group and taught the students myself. The design research experiment could be started when all preparation in the first phases of the design research was done.

The design experiment was conducted in the first three weeks of August 2008 (the first semester of the grade 4 in the academic year 2008/2009). At that time, we had received input from the try out in the preparation for the experiment, so that the first refinement or revision was already done before the design experiment was conducted. The new or revised Hypothetical Learning Trajectory, henceforth HLT II, was used in the design experiment phase as a guide for the teacher and researcher to do the teaching, interviewing and observing. We also did a daily retrospective analysis after we observed the students' learning process in this phase. However, there were likely to be chances to refine and revise the Hypothetical Learning Trajectory based on the daily retrospective analysis.

Next in the retrospective analysis phase, we analyzed the things that happened in the design experiment. The retrospective analysis was done in the period of September to November 2008. Everything that happened in class (see from the video recording) was analyzed. In this phase, we compared the Hypothetical Learning trajectory with the learning of the students in the class. We analyzed whether or not our goals of the design could be achieved. The struggles and the strategies of the students were also described. By the analysis, we could answer the research question that was posed. At the end, we had an analysis that was used to refine the Hypothetical Learning Trajectory II for a suggestion to have the betterment implementation of learning fractions.

3.2 Reliability and validity

Reliability has to do with quality of measurement [http://www.socialresearchmethods.net/kb/reliable.php]. Reliability is divided into two: internal reliability and external reliability. Internal reliability refers to the reliability within a research project [Bakker, 2004]. In this design research, we improved our internal reliability by discussing the critical moment in some episodes in the design experiment with our supervisors and colleagues.

External reliability means that the conclusions of the study should depend on the subject and condition, and not on the researcher [Bakker, 2004]. A criterion for virtual external reliability is the trackibility [Gravemeijer & Cobb, 2001; Maso & Smaling, 1998 in Bakker, 2004]. In order to make the readers able to track the learning process of the researchers and to reconstruct their study, during the design experiment, we took care of the data recorded. We provided two video cameras in the design experiment phase: one stand camera that was placed in front of the class, so the learning process of the whole class including social norms and discussion about the Mathematics in the class was recorded. Another camera was a still camera that focused on a group of students that could be used to see the development of mathematical learning of individual student. Besides that we also observed everything in the classroom and made notes in the observation sheet. The result of the interview and teaching activity with the students and the teacher would be transcribed and analyzed.

Validity is also divided into two: internal validity and external validity. The internal validity refers to the quality of the data collections and the soundness of the reasoning that led to the conclusion [Bakker, 2004]. To take care of the internal validity in this design research, we tested our conjectures that were generated at specific episodes, the pre-test and post-test that were given to students and students' worksheet and homework in the retrospective analysis phase.

External validity related to generalizing. In other terms, the external validity is the degree to which the conclusions in our study would hold for other persons in other places and at other times [http://www.socialresearchmethods.net/kb/external.php].

3.3 The description of subject of the research

Based on the basic curriculum of elementary school, the students had been taught about the introduction of fraction in grade 3. However, this design research was conducted in grade 4 (the first semester of academic year 2008/2009). This was conducted at an elementary school in Surabaya, East Java, Indonesia.

Hence, first of all, we needed to consider the standard competency of students of grade 4 and also an overview of the standard and basic competency of students of grade 3 related to the domain of this research.

Below are given the Indonesian National curriculum for grade 3 and grade 4 of elementary school.

National curriculum about the domain of the research in grade 3:

Standard Competency	Basic Competency
Number Understanding simple fractions and its use in solving problem	▪ Know simple fractions ▪ Compare simple fractions ▪ Solve problem related to simple fractions

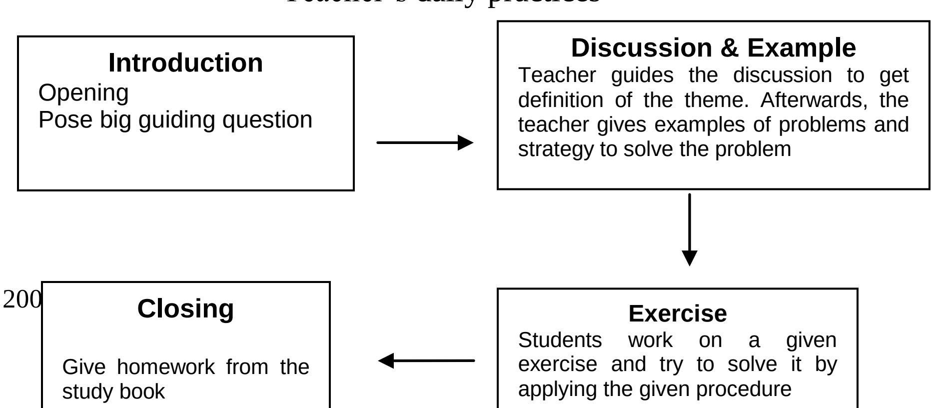
National curriculum about the domain of the research in grade 4:

Standard Competency	**Basic Competency**
Use fractions in solving problems	▪ Explain the definition of fractions and ordering fractions ▪ Simplify many kinds of fractions

The research experiment was conducted at the At-Taqwa elementary school (SDIT At-Taqwa) in Surabaya, East Java, Indonesia. SDIT At-Taqwa is a school that has high motivation to develop. This school had not known about Realistic Mathematics Education (RME) beforehand, which is why they were very welcoming and provided a class to be the subject of our research. The research was carried out in grade 4 with students of At-Taqwa Surabaya. There were 21 students in class IVB. For students in grade 4 (9 years old) of the At-Taqwa elementary school, they learned fractions in the second semester of grade 3. They had been introduced to fraction and they studied relations among fractions (comparing fractions) and simple operations with fractions (addition and subtraction of fractions). The mathematical theme was taught through abstract ways of learning, so the students were trained to use procedure to solve fractions problems.

Based on the teacher's information and the observation data in the try out session, first of all, the teacher posed a big guiding question, then gave every student an opportunity to answer that question. From the student's answers, all participants in the class could draw a conclusion about the theme that would be studied. The teacher informed us about his experience in teaching fractions; he used an apple as a model the first time he taught fractions and he spent four meetings (4x60 minutes) to discuss fraction (comparing, addition and subtraction of fractions). After these four meetings, students should be able to solve operations with fractions with equal and unequal denominator. From the observation, most students used procedural's strategies that were given by the teacher. Students were trained to master the procedure of solving fraction problems. For instance by did cross multiplication in comparing fractions. Below is the scheme of the teachers' daily practices:

Teacher's daily practices



IV. Local Instructional Theory

Based on Gravemeijer, one may work towards the goal of design research in combining two methods through developing local instruction theories and theoretical frameworks that address more encompassing issues. Local Instruction theory consists of : goals of the research, formulation of the research questions, conjectures about possible (mathematical) learning processes (conceptual and instrumental), conjectures about possible means of supporting that learning process, potentially productive instructional activities, envisioned classroom culture and a proactive role of the teacher. This conjectured local instruction theory is modified and revised based on retrospective analysis of the teaching experiment. Learning goals for students also is a component of the conjecture local instruction theory [Gravemeijer, 2004].

Before we made instructional sequence activities, we studied literatures and books with prior research in fractions and proportion. To complement some literature studies that we read, we decided to make sequence activities of learning fractions with fair dividing and measuring context with the paper bar, the string rubber bands and the number line as models. We chose the paper bar and the string rubber bands and measuring context, because those are not too far different from the number line, since based on Freudenthal (1973), the number line is the most valuable tools to teach arithmetic.

4.1 The Goals within this design research

4.1.1 The research goal and the research questions

The goal of the research is to develop students' understanding on fractions, the relation among fractions such as the equivalence of fractions and comparing fractions as well as the simple addition of fractions.

In the sequence of activities in this design research, we shall use certain contextual situations and models for learning fractions such as the paper bar, the string rubber bands and the number line that related to the context. Underneath, we raised a research question:

1. What roles do contexts, and models that are related to them, play in the teaching and learning of fractions?

Besides that, we believed that sequence of activities on the basis of RME indicate different ways of learning activities. In the RME, there are more discussion among students and between teacher and students. The teacher should guide the students construct their idea in learning. Thus, this raised another research question:

2. How do the social norms in teaching and learning develop within the sequence of activities?

4.1.2 Mathematical goals

- Determine the relation between fractions.
  Obtaining the concept of comparing and equivalence of fractions within the activity

- Determine the common denominator.
  Obtaining the common denominator logically within the activity
- Determine Addition of fractions.
  Obtaining the idea of solving addition of fractions with like and unlike denominator

4.1.3 Learning Goals for Students

Mathematical learning goals for the students are formulated below:

In specific domain (denominator up to 12) and through measurement activity within contextual situation:

- Students will be able to compare fractions and find the equal fractions
- Students will be able to find the common denominator of fractions
- Students will know how to solve addition of fractions

4.2 Conjectures about possible (mathematical) learning processes (conceptual and instrumental)

Through this design research, we conducted sequence activities that develop the concept of fractions: the relation among fractions (comparing and equivalence of fractions) and simple operation with fractions such as addition of fractions. The common denominator was also investigated in addition to those concepts. The students explored the common denominator through the activities instead of by introducing it directly in procedural ways.

For most students in grade 4, they define fractions as a part of a whole, because that concept is the easiest concept to be introduced in early learning fractions. When a thing or unit is divided into equal parts, the number expressing the relation of one or more of the equal parts to the total number of equal parts is defined as fraction. In this design research, we emphasized a shift from mastering the procedure to "understanding". To know the understanding of the students about the given concept, the teacher explored the students' reasoning in solving problems. Based on that, we conducted as our activity, the fair division of a cake using the paper bar as the model of the cake and the activity of symbolizing the paper bar using the concept of a part of a whole.

Models played an important role in this design research. In our plan sequence of activities, we used the paper bar, the string rubber bands and the number line as models. The paper bar folding was used in some of the first sequence activities. The string rubber bands were used as generalizing model of the paper bar folding. Then the number line model could be used as an abstraction of both previous models, the paper bar and the string rubber bands.

The paper bar was used as a model here because initially it was close to the contextual situation given: a long bar cake, we called it "Lapis" that should be divided into several number parts. This can be illustrated as some paper bars that must be folded into several numbers. The size of the paper bar should be equal to the size of the cake, because it made the model make sense for the student. The paper bar is a very good model to develop students' understanding of fraction relations or simple operations with fraction such as comparing fractions, equivalence of fractions and addition of fractions. With the paper bar, students could support their reasoning about the equal fractions, for example to find the equal fractions of ½; the conjecture is that they fold the paper into two and have one part be ½. They can also do it by folding the paper into 4 and having two parts of or in fraction notation is 2/4 as ½ of 4 parts.  . Beside the equivalence of fractions, students can use the bar model of the paper's shape for reasoning about comparing fractions.

Another task for students related to the paper folding activity is folding the paper bar into odd numbers such as 5, 7, 9, etc. I believed that students struggle with folding the paper into those odd numbers. To face this struggle, we introduced the string rubber bands as generalizing model of the paper bar model. To give an overview, below we give a picture of string rubber bands to illustrate what we mean:

The string rubber bands were introduced in the different context as before. We used the context of the cone cap game that is usually played on the Indonesian Independence day in which we need the string rubber bands to support the game. The string rubber bands model helps students to divide a length into odd numbers since the rubber bands can be stretched, for example: previously students struggled to divide a length into seven parts; now they can use seven string rubber bands and stretch them as long as the size of the thing that they want to divide. That is why in the previous explanation, we said that the string rubber bands model was as a generalized model of the paper bar model, because we could solve the division of length into odd numbers. The string rubber bands model also helps the student to reinvent equal fractions. The idea is for example to divide a long thing into two by using two string rubber bands. If the students use two string rubber bands to divide a long thing into two, it will be difficult to do since the string rubber band is not stretched enough. To solve it, they can use four rubber bands and use two parts or 2/4. It means that students will have an idea that ½ is equal to 2/4.

4.3 Conjectures about possible means of supporting that learning process

Below, we describe the tools and their imagery in the learning activity:

Tool	*Imagery*	*Activity*	*Potential Mathematical Discourse Topics*
The Paper bar		modeling dividing cake by clever folding	
The symbolizing of the paper folding	part of a whole	Give notation in the paper bar	symbolizing fraction
The paper folding bar	Explore the relation among fractions, simple operation with fractions, and the common denominator	the cake & shopkeeper context'	Relation among fractions and simple operations with fractions
The string rubber bands	Generalizing tool	measuring larger thing	The relation among fractions
The number line	Abstracting tool	Discuss the relation among fractions	The relation among fractions

In order to motivate the students to follow the sequence activities, the contextual situations were developed. The first contextual situation for students was the dividing the cakes context. The teacher told that she is challenged by her father to divide the cake fairly. Underneath the cake, we put a paper bar of the same size as the cake. The cake and the paper bar under it were brought to the class. Since we just brought one cake to be demonstrated, we challenged students to have another idea to divide the cake. Hopefully, the students would realize that there is a paper bar under the cake that can be folded. If they did not realize it, the teacher could show the paper bar to students and through a discussion, they would do the paper folding activity. Below is the picture of the cake, "Lapis Surabaya", that is familiar to students.

The modeling processes are conducted in this design. In the learning and teaching process, models play an important role. "Modelling" is a process in which a model is initially constituted as a context-specific model of a situation [Gravemeijer, 1994]. Emergent modeling was a part of the three instructional designs of Realistic Mathematics Education (RME). It is connected to developing more formal mathematical knowledge. Emergent modeling indicates a long process that covers informal forms of modeling and it generates the kind of mathematical knowledge that problem solvers need to construct mathematical models [Gravemeijer & Bakker, 2006]. In the reinvention principle, the idea is that the students construct models for themselves and this model serves as a basis for developing formal mathematics knowledge. To be more specific, at first a model is constituted as a context-specific model of situation. Later on, the model changes character, it becomes an entity on its own. In this new shape, it can function as a basis, a model for mathematical reasoning on a formal level [Gravemeijer, 1994]. For example: sharing a pizza constitutes a situation that generates fraction, a circle is the context-specific model as it provides an image of the pizza in the sharing process. In short, the *model of* informal mathematical activity develops into a *model for* more formal mathematical reasoning.

In the first activity, students are asked to divide the cake by folding the paper bar under the cake. In this case, the paper bar is *model of* the situation of dividing the cake. Beside this contextual activity, students were also given the chance to explore more contexts. They were given a place for sharing their own context, for example by giving them a paper to draw their thinking or their own context. Guidance is always needed especially to bring them to the bar model. Students used their drawing of a bar as a way to solve fraction problems and help them reason out their answers. In other words, drawing bars became the model for thinking and reasoning.

Another tool that was used in this activity is the string rubber bands as generalizing tool, since this tool can be used to divide the cake even into odd numbers that can not be done with paper folding.

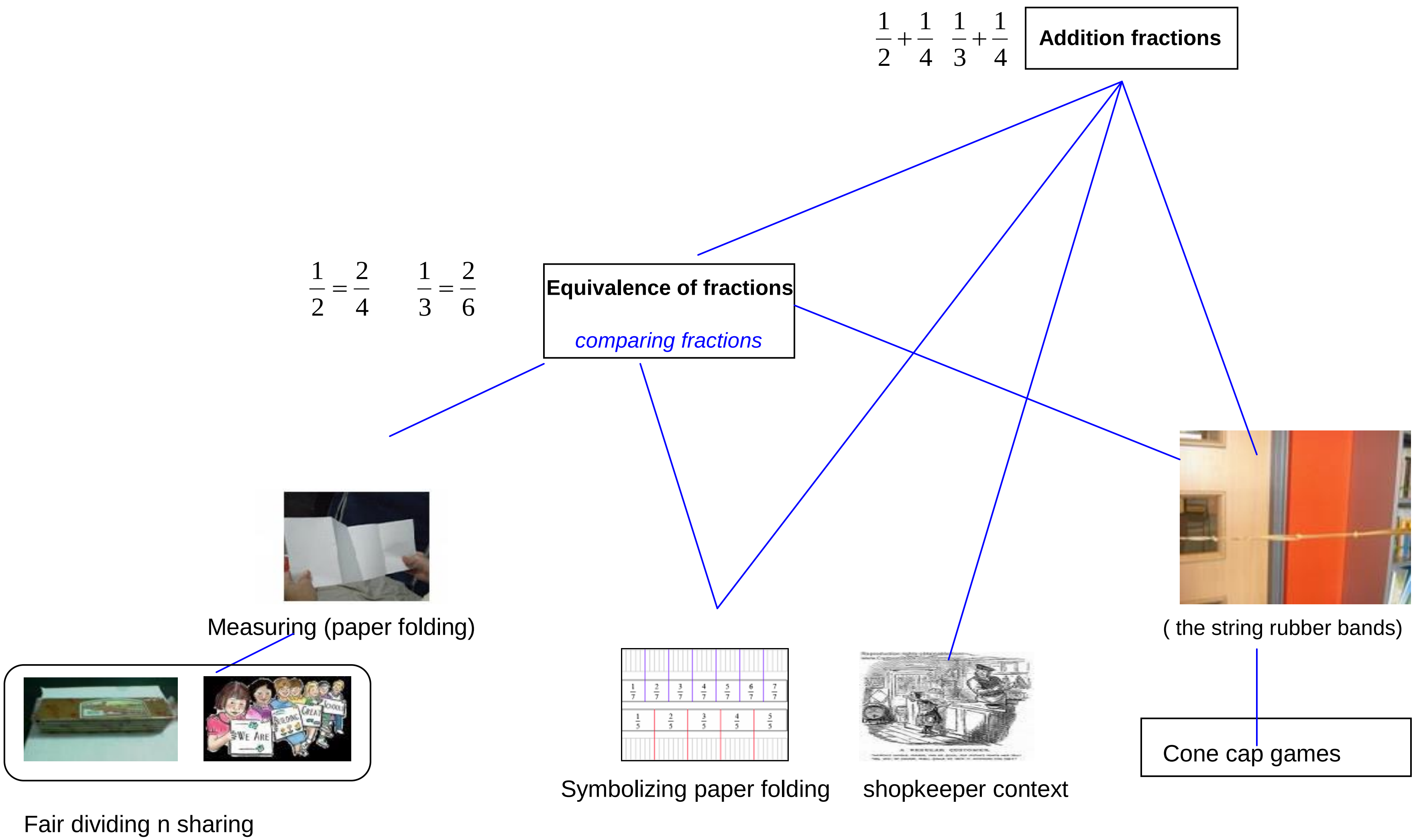
In the next plan activity, students symbolized the paper folding with the fraction notation. From symbolizing the paper folding, students are brought to exploring fraction relation on the number line. They can use the number line model to discuss and reason about relation among fractions.

In this design research, we intertwined the concept of fraction as part of a whole and measuring concept. The cake and measurement method are close to the paper bar as *model of* situation and the drawing bar as *model for* reasoning to solve the comparing and adding of fractions problems.

4.4 Learning Lines I

In the first phase of the design research, preparing for the experiment, we already made learning lines I. We called it learning lines I because that is the first learning line that we made and would probably need refinement before or after the research experiment. Before we started the research experiment, we tried out our design based on learning lines I to students in grade 3 (the grade before our research subject), since we did the research experiment in grade 4 (in the next semester after the try out activity).

Below is the Learning Lines I that was tried out first with students:


$\frac{1}{2}+\frac{1}{4}$ $\frac{1}{3}+\frac{1}{4}$
Addition fractions
$\frac{1}{2}=\frac{2}{4}$ $\frac{1}{3}=\frac{2}{6}$
Equivalence of fractions
comparing fractions
Measuring (paper folding)
(the string rubber bands)
Cone cap games
Symbolizing paper folding
shopkeeper context
Fair dividing n sharing

There were some changes and additions in the sequence activities based on the result of the try out activity. One of them was adding the 'number line' activity to the math congress of the activity of symbolizing the paper folding. This idea came from the discussion between me and the supervisors and it was also influenced by one of the problem in the pre test (fraction's train). In the try out of clever folding and the math congress (those two activities can be done in one meeting), students still found difficulty in exploring the equivalence of fractions as the basic for simple operation with fractions. So this exploration needs refinement of the equivalence of fraction's idea through the rubber bands activity that was held after the paper folding activity and the number line activity. In the try out session, the rubber bands activity worked out well and gave more help to the students to be able to find the equivalence of fractions. The students had more self confidence in answering the equal fractions' problem. For instance, in the try out of paper folding activity, Yogi struggled in finding the equivalence of a fraction. However, with the string rubber bands activity, he said, "it is easier for me with the rubber bands". In the try out session, it was found that the paper folding activity was also making sense for students. They used the paper folding activity for reasoning about comparing fractions, since it was easier for them to compare the different areas of the paper bar indicating fractions.

After all the activities that concerning equivalence of fractions, we continued with the shopkeeper context. This context concerns exploring the common denominator and used logical thinking and strategy and addition of fractions' activity which was not directly through abstract ways of learning. There was also an interview session and students were given exercise and homework about the equivalence of fractions. This was done in this order to make sure students were able to find equivalence of fractions. As well as the written post test, we also tested the students using the fraction card that after the written post test.

In the next part, we will show the revision of the sequence activities based on the result of the try out of learning lines I

4.5 The activity

4.5.1 Frame of sequence

- Construct a context which has a meaning of fair sharing or measuring
- Clever paper folding activity which has a meaning of fair sharing or measuring
- The symbolizing of the paper folding activity
- Math congress about the symbolizing the paper folding activity and **e**xploring the relation among fractions in the number line activity

- ❖ Refining the idea of equal fractions using the string rubber bands as model (The rubber bands activity)
- ❖ Math congress of the string rubber bands activity
- ❖ Expand to the idea of a common denominator through the shopkeeper context
- ❖ Math congress about the common denominator activity (shopkeeper context)
- ❖ Develop the context of the cake shop to explore the addition of fractions
- ❖ Math congress about the activity of adding fractions (cake shop context)

As well as the sequence activities above, pre-test, post-test and homework were given to students. The homework was given since students are used to get homework for mathematics subject every week. Interviews with the teacher and some students were done after the sequence activities to know their opinion and remarks about the design research. Below is given the learning lines II based on the sequence activities described above.

4.5.2 Learning Lines II (For the design experiment)

Learning Lines

$\frac{1}{2}+\frac{1}{4}\quad\frac{1}{3}+\frac{1}{4}$

Common denominator &

Addition of fractions

$\frac{1}{2}=\frac{2}{4}\qquad\frac{1}{3}=\frac{2}{6}$

Equivalence of fractions
Comparing Fractions

Measuring (paper folding)

(string rubber bands)

Fair dividing n sharing

(Symbolizing the paper folding) and extended to the number line activity

(Cone cap games)

(shopkeeper context)

4.5.3 Conjecture students' learning process

Based on the frame of sequence activities and learning lines II that was described in the previous part, next we will describe our conjecture of the students' struggles we anticipated and the conjecture of strategies that were used by students in the sequence activities.

The activity	Students' struggle	Students' strategies
Clever paper folding activity	- Using existing paper folding to be folded to some larger number (for instance: fold the paper into four from paper which is folded in two) - Fold the paper bar in an odd number (except for three, probably they still can do it)	Using paper bar that folds into two (half paper folding) as the anchor for the larger folding
The activity of symbolizing the paper folding	Understanding non unit fractions	- Give fractions' symbol by ordering the fractions - Symbolize each square in the paper folding with the unit fractions
Exploring the relation among fractions in the number line activity	Conflict in deciding the position of fractions in the number line	Use the definition of a part of whole and remember the activity of symbolizing the paper folding in order to explore the relation among fractions in the number line
Refining the idea of equal fractions using the string rubber bands as model (The rubber bands activity)	Achieving the mental strategy of finding equal fractions	- Use more rubber bands to divide a larger thing (for example: use four rubber to divide into two) - Use odd number of rubber to divide a thing into odd number
Expand to the idea of a common denominator through the shopkeeper context	Find the common denominator by using logical thinking and strategy through the shopkeeper context	- Multiply the denominator of each fraction - Use the paper folding as the models to explore the mental strategy of finding the common denominator
Develop the context of the cake shop to explore the addition of fractions	- Conflict with the idea of addition natural number (for example: 2/3+1/4=3/7) - Find the common denominator of the fractions - Find the equal fractions	- Use the idea of equal fractions - Use the common denominator

4.6 Logical thinking and reasoning

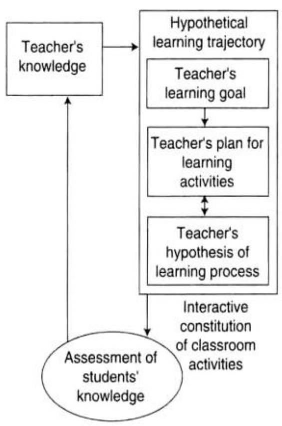
In the first sequence activities, the teacher showed the paper bar under the cake. Here, the logical thinking of the students was that they imagined that the paper bar that they could fold is as the cake that will be divided. Students might be facing difficulty in folding the paper into an odd number. In addition to find the total of two cakes, students used their previous information about the equal fractions. The second contextual problem is about helping the shopkeeper makes one measurement tool. The first logical thinking of the students is to have the common denominator of

2,3,4,6. For some students, to have this kind of logical thinking and reasoning, they used trial and error and other students might recognize it from the biggest number and be convinced that it can be divided into other numbers.

For the preparation of making string rubber bands to hang up the cone cap, students discussed and found the idea of the equal fractions. They found it difficult to use two string rubber bands to divide a line into two. As it was impossible to use two string rubber bands to have two line areas, because it was not stretched enough, students used four parts of the string rubber bands which means that they had the understanding that to divide the thing into two and notate it ½ in each part can be seen as 2 parts from 4 parts. This activity encouraged the logical thinking that to have ½ part of a thing can also be seen as 2/4, 3/6, etc of that thing.

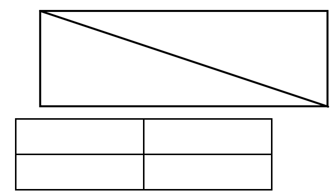
4.7 Hypothetical Learning Trajectory

Based on Simon, M and Tzur, R (2004), the Hypothetical Learning Trajectory henceforth HLT is a vehicle for planning the learning of a particular mathematical concept. The creation of the Hypothetical Learning Trajectory is based on the previous understanding of current knowledge of students.



Simon (1995)

Based on the description above, below is the HLT the Design Research. We called the HLT in this part as the HLT II. We give detailed information of The Learning Lines II after some refinement based on the result of the try out activity. The plan of learning activity was conducted in nine to ten days. Below is the detailed explanation of each day's activity:

No	The title of activities and the learning goal	Activity	Explanation behind the activity	The conjecture of students' strategies and the expectation
1	***The Clever paper folding activity which has meaning of fair sharing or measuring*** Learning goal: Students will explore the equivalence of fractions within paper folding activity. After the paper folding activity, math congress is hold in order to share students' strategy in fold the paper.	First of all, Teacher tells that she is challenged by her father to divide the "Lapis Surabaya" fairly. Students are asked to divide the cake by fold the paper bar below the cake fairly. The paper bar under the cake is shown in front of the class by the teacher. Based on that situation, teacher asks students' help to solve the problem. Below is the picture of Indonesian's cake that is familiar to students.  There will be some problem given to the students. They are asked to divide the cake based on the teacher's father demand. Some paper bar represent the cake will be folded by students. 1. Divide into 2 2. Divide into 4 3. Divide into 3 4. Divide into 6 5. Divide into 8 6. Divide into 5 7. Divide into 7 Students will work in group of two or three to have discussion between them. After they do all the task of paper folding, they will share their experience in math congress I. Worksheet is also given to students to describe the way how students folding the	This first contextual activity will guide the students to do paper folding. The idea is that students fold the pieces or parts of paper bar, all have to be in equal size. With this activity, students will be challenged to do "clever folding". Students will understand that dividing into a number means folding the paper into a number. First of all, students are asked to divide the paper bar into two, they will easily to do that. Next they are asked to fold the paper bar into four. In folding the paper into 4, they will do it through the previous folding paper strategy (fold the paper into two). They will fold the two part paper into two again, so they will have four parts. Students will investigate the relation among fractions by this activity, for instance, they can divide a paper bar into 4 parts by first finding "1/2" and then dividing each half into two parts. They will see that 1/2 will be equal to	There are some papers bars given represent cake to be divided fairly. Since the cake is long cake and it can be represented with long paper bar, the fourth grade students probably will use some tools that are familiar for them such as ruler to measure and divide the paper fairly. Let students do what they want first. To face students who use tool to measure and divide the paper, the cake must be made so that it would difficult to be divided use ruler for example. The different strategy will be used by students, they can divide the cake into several shapes such as  If we face this condition, we can anticipate with using a cake that usually cut in slices so they will not cut it diagonally like the probably happened above. Other conjecture is that there are students who fold the paper to have fair division. If we found this strategy, we can show it to other students with give advice that it will be handier to do, beside that students can train to have clever paper folding through

<table>
<tr>
<td></td>
<td></td>
<td>paper bar. Interviewing and guiding students also can be done in this activity.</td>
<td>1/4 +1/4 or 2/4.
The task includes dividing into five and seven. Although our conjecture is that students will find difficulty in dividing the paper bar to those numbers, we still put it as a task to let them realize that dividing a paper bar to five or seven are more difficult than to other numbers.</td>
<td>that strategy. Through taken as shared and use clever paper folding as strategy to divide the cake fairly, students do paper folding activity to divide the paper based on the question in students' worksheet. Discussion in group to have clever paper folding held. Students also will describe their strategy in the student's worksheet.</td>
</tr>
<tr>
<td>2</td>
<td>The activity of symbolizing the paper folding

Learning goal:
tudents will be able to symbolize the folding paper bar with fraction's symbol, not only with unit fraction but also with non unit fraction.</td>
<td>Symbolizing the paper folding is the next step after paper folding activity. Students already have some paper bar that already fold into some number. Then students are challenged to give fraction notation on all those paper.

.</td>
<td>The activity will guide the students to do symbolizing. The unit and the non unit fractions will come up trough this activity. Let them symbolize and explore the fractions themselves. For the non unit fractions, there will be a discussion related to the number line of fractions. The discussion about the meaning of a fraction that they write somewhere in the number line. By this activity, students will not only understand the unit fractions. They also will be exercised with non unit fractions and ordering fractions</td>
<td>Probably there are some student's still use unit fractions instead of non unit fractions. Every part of the paper bar is symbolized use unit fractions. For example: the paper bar that is divided into 3 , they will symbolize as below :
<table><tr><td>1/3</td><td>1/3</td><td>1/3</td></tr></table>
For students who use unit fraction, we can guide them to the idea of non unit fractions. By the non unit fractions, the discussion will be brought to the idea of the number line of fractions.</td>
</tr>
<tr>
<td>3</td>
<td>Explore the relations among fractions in the math congress I about</td>
<td>Teacher held class discussion. Teacher will guide the discussion. Some groups of students are chosen to share their idea and teacher can</td>
<td>Our conjecture is that there are students who write unit fraction to every part of paper</td>
<td>The expectation that hopefully appeared from the number line activity, students will posit the fraction in the number line. With</td>
</tr>
</table>

<table>
<tr>
<td></td>
<td>the activity of symbolizing the paper folding and the number line activity

Learning goal:

Students will share their idea and their experience in giving notation in the paper folding and also discussed the meaning of non unit fraction and comparing fractions in the number line.</td>
<td>ask to other students about their friend’s strategy, so there will be a discussion in the class. There is also a discussion about non unit fraction and its meaning through number line model. The number line that is given is not the drawing of number line on the blackboard, but we use a rope and some fraction paper that can be hanged on the rope.</td>
<td>folding like below:

<table><tr><td>1/3</td><td>1/3</td><td>1/3</td></tr></table>
And there are also students who write fraction notation as below:

<table><tr><td>1/3</td><td>2/3</td><td>3/3</td></tr></table>
For the second notation, there will be a discussion in the number line model and discussed, for example: 2/3 is the name of just the second part or that is the name of the first two parts together? The explanation can be drawn in the number line. Students will put the fraction in the number line and explain the meaning of it. Through this activity, students do not only understand the unit fraction, they are also trained with non unit fraction. The different strategies from students will appear in the class discussion. Teacher will guide students to get the idea of equivalence fractions from paper folding activity.</td>
<td>the number line, students will show the equivalence of fractions and comparing fractions. Students can reason that the bigger fraction, the more to the right is its position.
For the equivalence of fractions, the fraction cards that are put in the same position are equal fractions.</td>
</tr>
</table>

4	***Refining the idea of equal fractions using the string rubber bands as model (The rubber bands activity).*** Learning goal: Students will refine their understanding about the equivalence of fractions through the rubber bands activity.	The New contextual situation derived about game of 17 August to celebrate Indonesian' independent day. The game that will be held is the cone cap game. Before student play the game, they have to help the teacher prepare everything. Below is the picture of the game to give overview  For the game, it needs to prepare fair distance of string for place for hang the cone cap. At that moment, there is no rope that can be used but teacher have an idea to use the rubber bands since she just see some kids play with string rubber bands. Teacher shows some of the big rubber bands and ask student to make string rubber bands based on how many parts of area that they want to divide. For example: with two rubber bands, they will divide the length of black board in two and put a hanger in the middle of it. Our conjecture is that they can't do it with two rubber bands to have ½ parts. Student will discuss and use 4 or 6 or 8, etc to divide the string. They shall come to the idea that ½=2/4=3/6=4/8 With three rubber bands, they will divide the string in three and put two hangers in every tied rubber band.	The new model is used in this activity that is the string rubber bands. This model can help students to prepare fair distance of the string rubber bands to put the hanger in it. With the characteristic of the rubber bands that can be stretched, it will help student to put the hanger of the cone cap with fair distance. With this new model, student also can solve in dividing a thing in odd number, for example if they want to divide a thing into seven, they can use seven rubber bands and stretch it if the thing is long. With this activity, it can help students to refine their understanding of equal fractions. The mathematical discourse that came up through this activity is operations with fraction that can also be done with denominator more than 12.	The student's strategy that probably come up in dividing into some number is that students will make a string of rubber bands, but of course the frustrating thing come up since it will be difficult for them if they have to divide the line into 2 for example since the two string rubber bands is not enough to be stretched that long distance. They shall find that they can use four; six, eight, etc then divide it into two. Probably the confusion came if students divide the line area into two; they will give one sign in the middle of the line to hang the hanger. If they divide into three, they can put two hangers in it. But since the mathematical idea shall come in the process of finding the fair area lines (the equivalent fractions), it will not really to be confused by the students

		Students will explore to divide the string also in 4 and 6.		
5.	***Math Congress II about the rubber bands activity*** Learning goal: Students will share their experience and idea of equal fractions within certain domain (denominator up to 12) and odd denominator	Teacher held a class discussion. Teacher will orchestrate the discussion. Some group of students will be chosen to share their idea and teacher can ask other students opinion about their friend's strategy so there will be a discussion in class. The activity can be emerged to other unit fractions	Students can share their knowledge related with equivalence of fraction through rubber bands activity.	In sharing knowledge, students show their previous experience when work with rubber bands.
6.	**Interviewing students and giving exercise**	Students are divided into two groups. They are interviewed by researcher about their experience and their finding of equivalence fractions. Beside that, students are asked to solve some problem with discuss together with their friends.	The interview and give exercise are held in order to make sure that students really get good understanding in equivalence of fractions as the basic of the next activity (activity of addition fractions).	Students will answer some question from interviewer and reason their answer.
7.	***Expand to the idea of a common denominator through the shopkeeper context*** Learning goal: Students will share their ideas and their experiences in the problem of helping the shopkeeper, related to the idea of the	The next contextual problem is appeared. The teacher continues her story about shopkeeper who sells long cake. The shop provides the long cake and sells it in small part. Teacher tells story that the shop is very crowded everyday. The shop sells 1/3, 1/2, 2/3, ¼, ¾ and 1/6 cake. In the previous activity, students already made some paper folding of the part of the cake above and the teacher show it to the shopkeeper. But, the shopkeeper asks whether he really needs all the strips as the toll to help the shopkeeper cut the cake? The teacher can also ask whether it is possible to	This contextual situation will guide students to the idea of common denominator. Students do not only work with unit fractions, they also work with non unit fractions. Part of the cake that is given with denominator 2,3,4,6.	By discussion, students face some common denominator from many fractions' combination. For example, if they have strip for ¼, they don't need strip for ½ (the idea is that 4 is the common denominator of fraction with denominator 2 and 4. If they have strips with fraction with denominator 6, they don't need strip for ½ and 1/3 (the idea is 6 is the common denominator of fraction with denominator 2 and 3). By the last questions, we will guide students to common denominator 12 (for fraction with denominator 2,3,4,6)

	common denominator	just make one strip as helper cutting tools. Let students discuss in a group and the paper folding can be provide if it is needed by students. Beside that, worksheet is provided to record the thinking process of students.		
8.	***Math Congress III about the common denominator activity (shopkeeper context)*** Learning goal: The Students share their knowledge related to the idea of common denominator	Teacher held a class discussion. Teacher will orchestrate the discussion. Some group of students will be chosen to share their idea and teacher can ask other students opinion about their friend's strategy so there will be a discussion in class. The activity can be emerged to other unit fractions	Students can share their knowledge related with the idea of common denominator.	Class discussion
9.	***Develop the context of cake shop to explore the addition of fractions*** Learning goal: Students will be able to solve the addition of fractions in a certain domain (denominator up to 12)	This activity is still related with the shopkeeper context. The cake shop sells two types of cake (chocolate and vanilla). The price of each both cakes is Rp. 12.000,-. There are so many customers who want to buy two parts of cake, for example: 1/3 chocolate and ½ vanilla 2/3 chocolate and ¼ vanilla 1/6 chocolate and ½ vanilla ¾ chocolate and 1/12 vanilla To make the shopkeeper work as fast as possible, students are asked to look for the total part of cake and the total price that should be paid. Let students discussed this problem in group. The worksheet is also provided to record their strategy in helping the shopkeeper.	This activity will explore addition of fraction.	There are some possibilities, one of them is that students directly count the price of every part of the cake, for example by calculate every part of the cake by the price and then add it. Other probable strategy is that students will find the common denominator first, then find the equivalence of its fractions and add those fractions wit like denominators. The price will be calculated afterwards by using the bar drawing or the paper bar to help the students.
10.	***Math congress IV about the activity of the***	Teacher hold a class discussion. Teacher will orchestrate the discussion. Some group of	Students can share their understanding and their	Probably there are still some students who struggle with finding the equivalence of

	addition of fractions (cake shop context) Learning goal: Students can share their understanding and their strategies to find the result of the addition of fractions.	students will be chosen to share their idea and teacher can ask other students opinion about their friend's strategy so there will be a discussion in class. The activity can be emerged to other unit fractions	strategy to find the result of addition fractions.	fractions of the non unit fractions. To face the struggle, teacher can use the bar as the model to help them solve the problem or remind them to the previous activity related to the equivalence of fractions.

4.8 Role of the teacher

In this design research, the first role of the teacher in the activity started by telling a story that motivated the students to do the activity (measuring). The given contextual situation was about dividing the cake makes sense to students and the way of teacher explains attracted students to do the activity. Before the students came up with the idea of the paper folding, they discussed a better way to use the paper bar.

The role of the teacher in this activity was not to transmit their knowledge to the students, but to orchestrate the discussion to bring up the idea of folding paper. A math congress is held as a space for students to share their ideas. In the math congress, teacher functioned as moderator and guided the discussion to bring students accept idea as “taken as shared”. The guidance was needed in order to explore the students’ logical thinking and reasoning, for instance, using the string rubber bands to have the idea of equal fractions and using the paper bar with the shopkeeper context to investigate the common denominator logically, while avoiding abstract ways of learning.

4.9 Envision classroom culture organization

The instructional activities were done in two types of classroom organization:

First, students worked in a group of two or three to discuss and solve the instructional activities. At that time, the teacher observed and interacted with the students in small groups. After students worked together perhaps 20 to 30 minutes, the teacher initiated whole class discussion of students’ interpretations and solutions (math congress). The teacher posed questions to guide students in the classroom discussion

Based on the explanation above, there were small group activities and whole class activities. The social norms of small group activity would become the topics for the math congress, including solving personally challenging problems. For instance, students shared their strategy of folding the paper into 4 and folding it again into two to get 8 paper folding. Explaining the personal solutions and listening, then commenting on the partner explanation were also part of the social norm of the small group activity. The different strategy of folding the paper could also be discussed and commented in the small group discussion. The discussion would attempt to achieve an answer or solution for the problem. The small group activity served as the basis for the whole class discussion.

4.10 Assessment

Assessment instrument included paper-pencil test and interview.

Interview instrument

The interview instrument consists of a pre-interview and a post-interview. This interview was conducted by the researcher. Some questions were given to students to examine the students' ability to understand fractions and its operation. The researcher gave the students as much as time as they needed to answer the question, with each interview lasting about an hour. The interview was conducted in small groups of students (8-9 students) with various ability, which means both better achieving students and low achieving students (the information about students got from my observation in the design experiment and it was also given by the teacher).

The process of the interview was videotaped, and then some of the recordings being transcribed for this thesis.

Paper pencil test instrument

Paper pencil tests included pre-test and post-test for students. The aim of the pre-test assessment is to see their initial knowledge from the previous activity that related to fractions. If students can give an explanation and answer the pre test correctly, they understand the meaning of fraction as a part of a whole and ordering fractions. These two things will be useful for comparing fraction and addition of fractions that will be explored through the sequence activities. The pre-test consists of three questions.

The post-test was given in order to measure the understanding of students after following the learning activities. If they are successful in working with the task in the post test, their work indicates a progress in understanding. The post-test consists of four questions with some sub questions in every question. The given questions were about students' understanding of the mathematical goal of the sequence activities: equivalence of fractions, comparing fractions, the relation among fractions in the number line and addition of fractions.

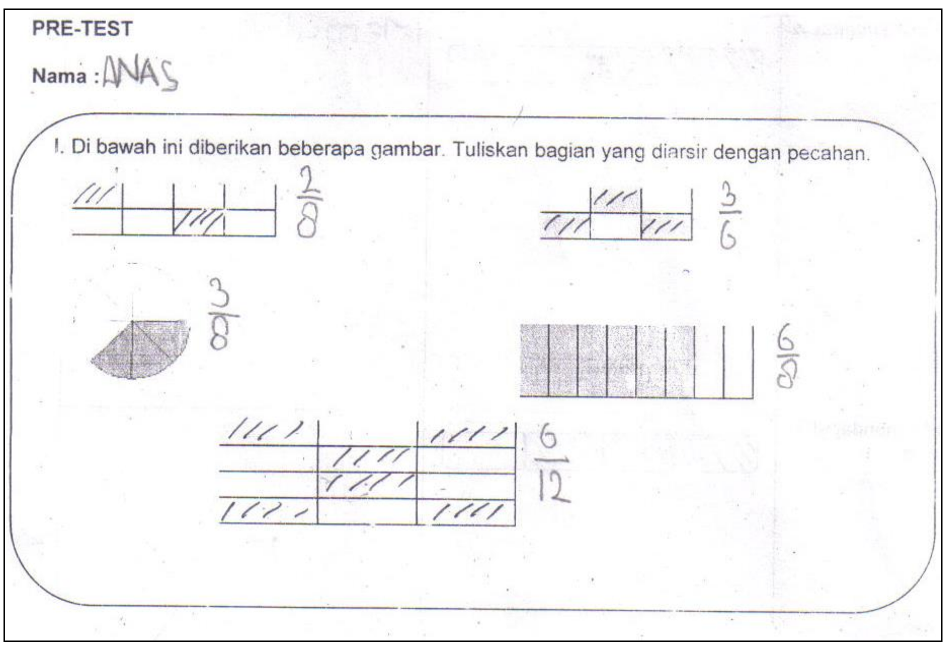
Both the pre-test and post-test are open problems. Afterwards, all the data from pre-test and post-test were analyzed. The pre-test was analyzed in order to see whether the students were able to follow the sequence activities. The post-test was analyzed in order to see the development of the students' learning process. The pre-test and the post-test that were given to students can be seen in the appendices.

V. Retrospective Analysis

5.1 Pre assessment of student's knowledge

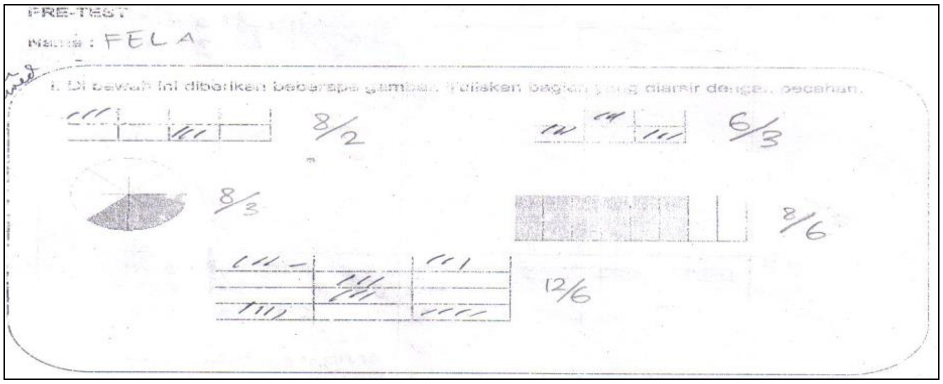
Based on the Indonesian curriculum, simple fraction is taught in grade 3. Based on the information from the students with whom we worked, their former teacher had introduced fraction and given a procedural way of solving simple operations with fractions. I believe that the students were familiar with simple fractions, but not all of them understood their corresponding meaning. To discover the pre-existing knowledge of the students before they involve a sequenced learning activity, a pre-test was given to the students. We observed whether or not the students were able to follow the activity. The pre-test was done in three problems to check students' understanding of fractions.

The first problem was giving the fractional notation for each shaded picture. 75% of the students could answer the problem correctly while the rest was not able to solve it. Quantitatively, we could infer that most students understood the concept of fractions as a part of a whole. Take as an example, as seen in Anas' pre-test in figure 5.1 below:



(Figure 5.1)

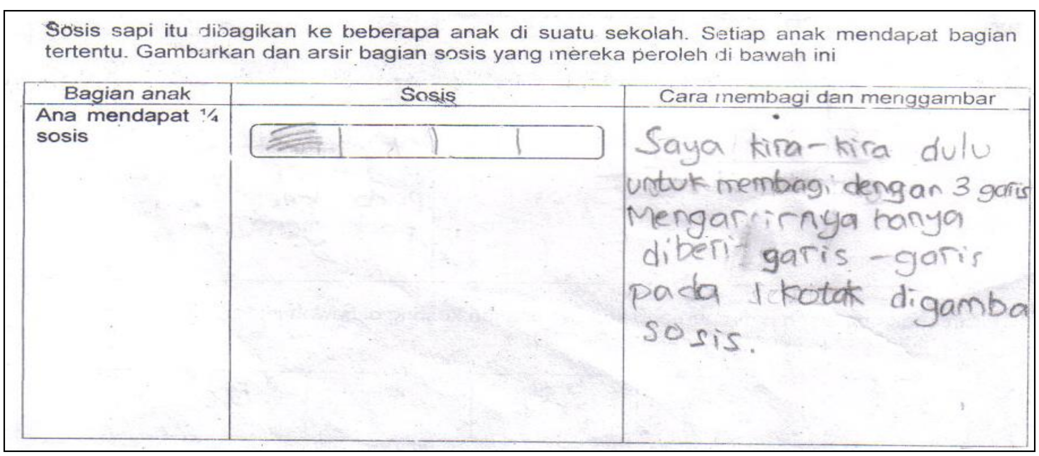
Students who could not give the fraction's notation correctly committed two different mistakes. Some of them wrote the fraction of a shaded picture as: $\frac{whole\ area}{shaded\ area}$. An example could be seen in Fela's worksheet in figure 5.2 below:



(Figure 5.2)

Besides the one as described above, there was also a mistake from a student in the last drawing. She represented the fraction in a shaded picture as: $\frac{shaded\,area}{unshaded\,area}$. She argued that the fraction for the last drawing (a rectangle consists of 12 small rectangle and 6 of them were shaded) was 6/6.

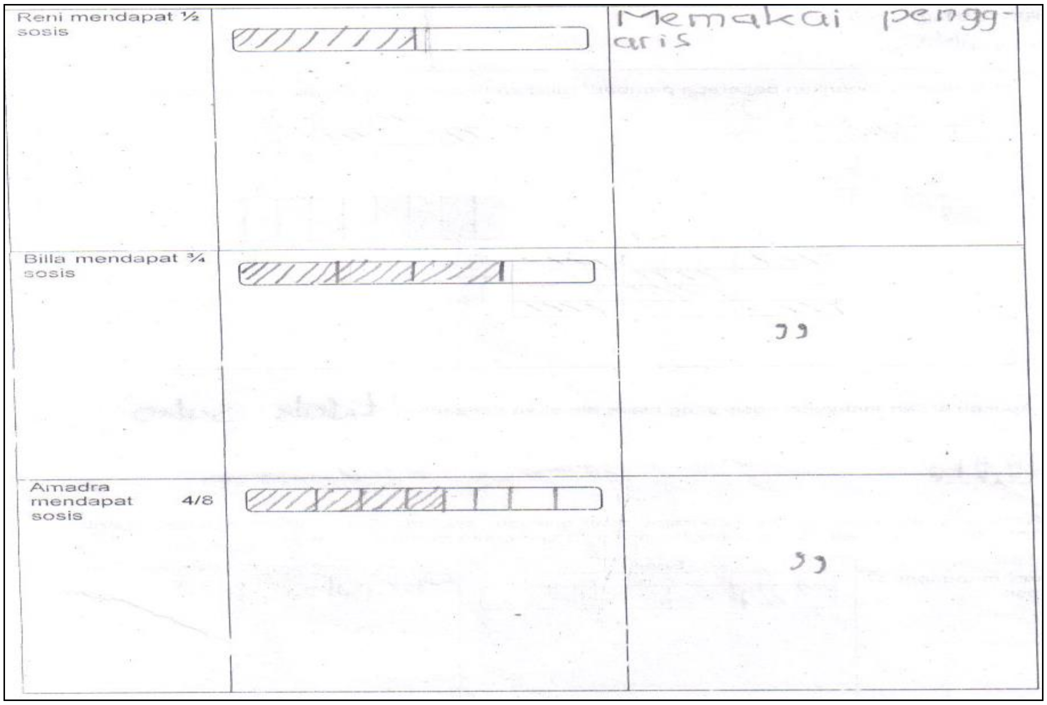
The second problem of the pre-test was testing the students' understanding about the meaning of fractions. The problem was about to make a fractional drawing on the bar and the students should also describe how they made the drawing. Most of them could show the fraction in their drawings. There were different strategies to describe their drawings for the fractions in a sausage bar; one of them was making an estimation of the drawing division.



(Figure 5.3)

(The translation of figure 5.3: *First, I tried to estimate the division with three lines, and then shaded one square of it*)

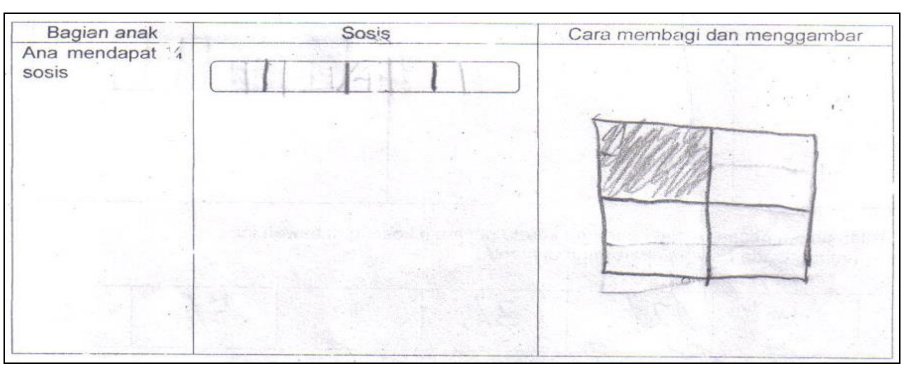
Other students explained that they had used the ruler to measure. Others had divided the sausage from the left side of the drawing (Syarif's strategy). Below Anas' strategy (figure 5.4) is shown.

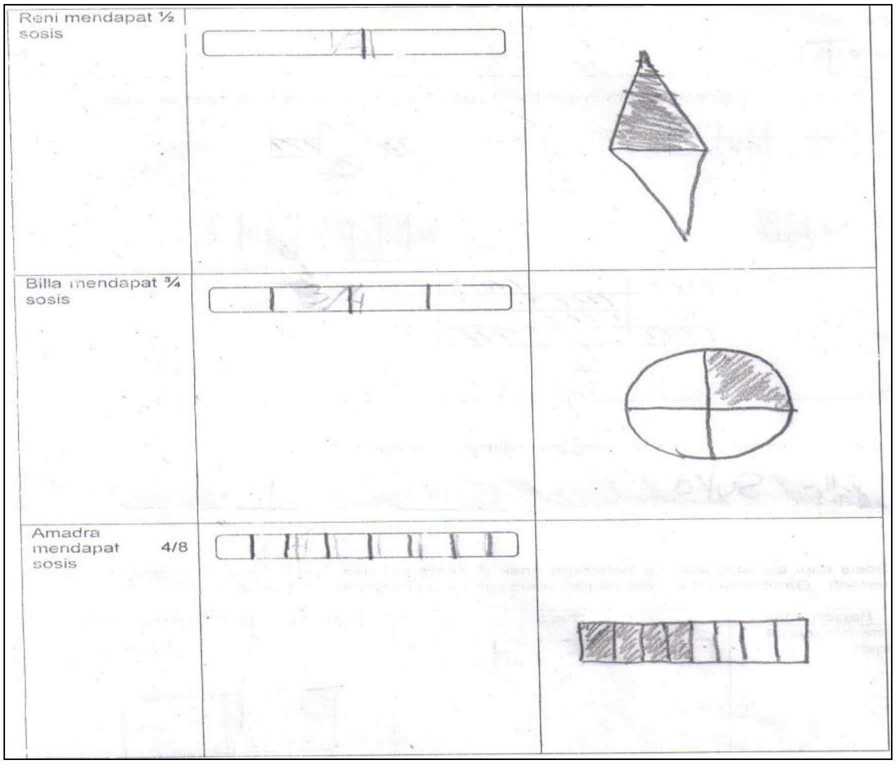


(Figure 5.4)

(The translation of figure 5.4: *Using ruler*)

Apart from the above strategies, there was also a student who changed the given drawing into one that he was familiar with. Thoni drew a square, parallelogram and circle to represent the fractions (see figure 5.5).

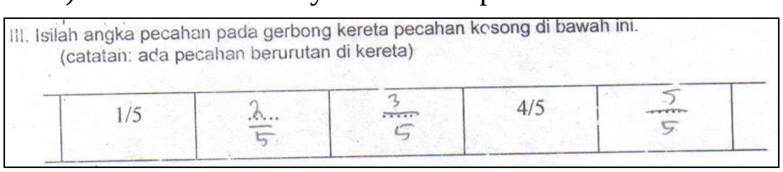




(Figure 5.5)

The rest of the students could not give a description of their strategy. Despite their obviously different strategies, they actually understood fractions as a part of a whole.

The third problem in the pre-test was about ordering fractions. There was a fractional train with some fractions in it. Students were asked to fill the blank chunked train by ordering it. As a hint, some fractions were given. The problem was related to one of our sequence activities, "the number line activity". Only two students (Anas and Kevin) could not correctly answer the problem.


III. Isilah angka pecahan pada gerbong kereta pecahan kosong di bawah ini.
(catatan: ada pecahan berurutan di kereta)

1/5	2/5	3/5	4/5	5/5

(Figure 5.6)

(The figure 5.6 is Alya's worksheet)

We gave the pre-test to students in order to see their prior knowledge about fractions. From the students' pre-test result, we could conclude that most students understood the meaning of a fraction as a part of a whole. In other words, the students' knowledge about fractions was sufficient to follow some of the first sequenced activities, such as symbolizing the paper folding and the number line activity that would lead to simple operations with fractions as planned.

In the next part, we will explain and describe the analysis of the result of our design experiment. We will focus on the topics that show interesting findings and are appropriate with my expectation of a certain activity as being a rich and fruitful activity, like the number line activity and the activity of exploring the common denominator in a non abstract way. I will also describe an unforeseen question by a student which was asked in the discussion, and then criticized and responded to by her friends. The challenges and struggles of students will also be described in the next part. As well as the good things, we will also discuss some activities that need to be revised in order to achieve the goals behind the activity. For example, while the paper folding activity and the rubber bands activity were great activities, we saw in the research experiment that they need revision.

5.2 The activity of symbolizing the paper folding and the number line activity as fruitful and rich activities

The number line activity is a new activity that was developed after a discussion with supervisors during the try out activity. Applying the notation to the paper folding, holding a math congress about it and the number line activity were fruitful and rich activities. We used a rope as the number line to motivate students to work with fractions. The activity of symbolizing the paper folding was the next step the students had to do after the paper folding activity itself. The learning goal of the activity of symbolizing the paper folding was to enable students to symbolize the folded paper bar with fractions symbol, not only with a unit fraction but also with a non unit fraction. Why do I believe that those two activities were rich ones? This was because in the activity of symbolizing the paper folding, we expect students to abstract and symbolize the unit and non unit fraction and to explore their understanding of comparing fractions.

The number line activity tests students' understanding of the meaning of fractions, the equivalence of fractions and the comparison of fractions. Beside that, I expect that this activity will reveal students' reasoning. From this research experiment, we will see the proof of my conviction. During the math congress about the symbolizing the paper folding and the number line activity, I expected the students to share their knowledge until the agreement was reached. Correcting, giving suggestions and criticizing each other appeared in the math congress. In this period, I expected the students to come up with some ideas of their knowledge of representing the fractions, for example that they came up with the unit and non unit fractions notation or that they might just shade the bar to represent the fractions.

In the number line activity, we expect that students will realize and reason that the bigger the fraction, the more to the right its position should be. If some fractions have the same position in a number line (henceforth the number line of fractions), they could conclude that those fractions are equal fractions.

On the whiteboard, we glued three different paper bars that had been folded into 3, 6 and 8 pieces and a rope with the same length as the paper bars. The choice of those folds was based on a hypothesis that by using a paper bar that folds into 6 pieces, students could explore fractions with denominator 2, 3, 6, and by using a paper bar that folds into 8 pieces, students could explore fractions with denominator 2, 4, 8. Before the students worked with the number line of fractions, we had tested their understanding of the meaning of fractions by asking them to symbolize the folded paper that was glued on the white board (This was also based on students' strategy in the activity of the previous day that is the activity of symbolizing the paper folding). During the activity of symbolizing the paper folding, students were given a worksheet to describe their discoveries in symbolizing fractions and comparing fractions.

5.2.1 The activity of symbolizing the paper folding in the classroom

Different strategies were described in the activity of symbolizing the paper folding by the students in the class. For instance, Rheina and Maria were discussing (as a group) and doing the task together. In their worksheet, they showed that they ordered the fractions on the paper bar that was provided. They also found the equal fractions for some symbolized paper folding. Through the math congress about the symbolizing of the paper folding, we could see the different strategies of the students in giving the notation of the fractions. The activity of symbolizing the paper folding was a basis for going to a more abstract learning (the number line activity), so the students' struggles with the meaning of fractions should be discussed and corrected before we started the number line activity.

The activity of symbolizing the paper folding and the number line activity were rich and fruitful activities; however, there were still some problems with the students' understanding of the meaning of fractions. Initially, it was necessary for the students to experience a fruitful discussion in the number line activity. My first question as a stand-in teacher in the math congress about the activity of symbolizing the paper folding and the number line activity was “Who can show 2/3 in the folded paper bar on the whiteboard?” Danti initiated by shading the folded paper bar on the whiteboard. Then Nouval (N stands for Nouval) came up with a different strategy. He wrote 1/3 on the first square and 2/3 on the second square as shown in the picture below:

(Figure 5.7)

When I tested Nouval, there was a problem with understanding the meaning of fraction, but later, Nouval’s problem could be resolved by Faiz’s help. I asked Nouval about the 2/3 on the paper folding as indicated in the transcript below, where Faiz, Researcher, and the students are encoded as F1, R, and S respectively.

R: Was 2/3 just the middle square or which one was it?
N: This was 2/3 (he points and shows 2/3 as the second square of the paper folding)
S: hah…what??..no..no
R: Were there any different opinions?
F1: Nouval, 2/3 was not just the middle square of the bar as you showed but all the first two squares, so all except the last one (explain while pointed the paper bar)
R: How about the last one? What fraction was it?
S and F: 3/3….one.

The first “taken as shared” of ordering and symbolizing the unit and the non unit fractions came from Nouval when he symbolized and ordered fractions based on my instruction, and came from Faiz also when he corrected Nouval’s understanding of the meaning of 2/3. Nouval and Faiz’s discussion in front of the class could be the initial knowledge of the meaning of fractions for the whole class to move on to the number line of fraction.

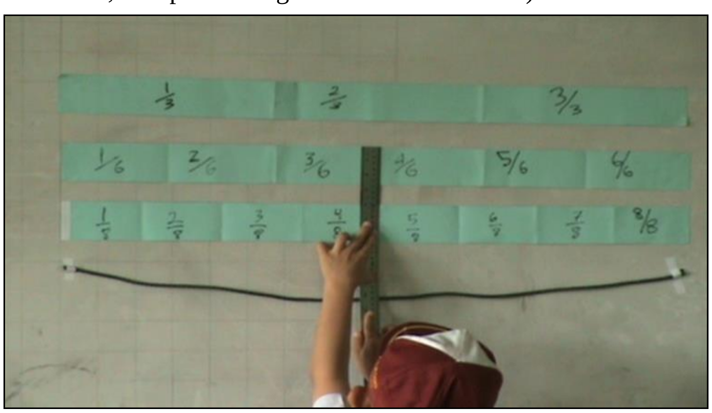
It was probably a good question to ask about the shading strategy used by some students, for instance, Danti's strategy that was explained before. Their strategy could still be appreciated because those students were still at the level of drawing. They needed more time to go beyond the level of abstracting or symbolizing. The discussion moved on to the next two fractions, which showed 4/6 and 7/8 on the paper bars, while Himmah filled the rest of the blank squares of the paper folding by ordering them.

5.2.2 The number line activity

After an agreement on the meaning of a fraction was achieved during the activity of symbolizing the paper folding, the next episode was the number line activity. We provided a blank rope as a tool to hang the fraction cards on rope. The three symbolized paper foldings were put above the rope. I began with a fraction that had not been done in the paper folding. There were some students who participated in sharing their ideas, to name them, A1 for Anas, B for Bella, A2 for Alya, H for Himmah, Rh for Rheina, M for Maria and F2 for Firda.

R: Who could show ½ on the rope?

A1: Here was ½ (He points to the rope and with the ruler, makes a line from the last border of 3/6 And 4/8; I help him hang the fraction card on it)



(Figure 5.8)

For the students in grade 4 (9 years old), a half and other unit fractions were familiar fractions. They were used to use a half as the anchor point. Then, how do they go about the non unit fractions? To answer this question, we gave them a challenging task asking the position of 3/6 on the rope. The students agreed about the position of 3/6 as transcribed below:

H: On the half (while pointing to the last line of square 3/6)

R: Himmah said that 3/6 was in the position of 1/2., what do you think?
S: (shouted) yes, that's right…
H: 3/6 on the rope was from the left point of the rope to the point of 3/6.
R: Okay, why it could be in the same line, the half and 3/6?
B: in the same straight line.
R: How it could be?
B: Because those were equal fractions
R: Were there other fractions that could be put on the half position?
Rh: Of course there was…4/8.
R: How it could be in the straight line?
R: Was there anyone who could conclude something?
M: Because those were equivalent
F2: yeah ½ was equal to 3/6 and equal to 4/8

That fruitful result from the number line activity was achieved after there was a discussion among the students to look at the problems that came up during the previous discussion, for instance a conflict about understanding of the meaning of the fraction symbol in the paper bar. Another question was about positioning the non unit fractions (showing where 6/6 was). Agreement was achieved after Anas shared his finding that 6/6 was equal to 1, meaning that it takes up the entire rope.

Another fruitful investigation that made the number line activity a rich activity was comparing fractions. First of all, the teacher asked the students about some different fractions' position on the rope. As an example, the teacher asked them to show 1/3 or other fractions on the rope. Maria easily showed and marked of 1/3 on the rope. Then, I created a conflict among the students to determine "which was bigger between 1/3 and 3/6 (note that 3/6 was the previous fraction on the rope)?". Some students answered that 3/6 was bigger and the rest claimed 1/3 was bigger. Students who argued that 3/6 was bigger than 1/3 reasoned on their finding as suggested in the transcript below:

R: Why was 3/6 bigger than 1/3?
M: Because 3/6 has more squares
R: How if you see it on the rope?
T: Because 3/6 was further than 1/3 (T for Thoni)
F1: 3/6 was further than 1/3 from the front
R: If I put 5/6 on the rope, do you think which was bigger between 5/6 and ½?
H: 5/6 because it was further and goes almost to the last edge of the rope

Maria's reasoning that 3/6 is bigger than 1/3 is because 3/6 has more squares. This seems to be simply correct because she argued that more shaded squares to show fractions indicated that a fraction is bigger than another fraction, but indeed mathematically incorrect. More shaded squares of a fraction do not account for the fraction that was bigger than other fractions with the less squares shaded. The students' reasoning in the transcript above shows their idea of working with the fraction number line. It was due to the fact that students remembered and understood the concept of the natural number line that the bigger the number is, the more to the right its position should be.

Next, we explored the equivalence of fractions through positioning ½ and 3/6 on the number line. As was said before that the number line activity was a fruitful activity, does not mean that there were no students struggled with positioning the fractions. Some struggled with positioning the non unit fractions on the blank rope. For example, before the students agreed with the position of 3/6 on the blank rope, Bella and Alya had a problem in deciding the position of it.

R: Who could show 3/6 on the rope?

B: mmh..mmhh I think here (She points the 3/6 on the rope as the middle of "3/6 square" on the paper folding)..wait..wait..3/6 was on the half position. But mmh no no..(Then she came with the first conclusion: 3/6 was in the middle of "3/6 square" on the paper folding)

(Figure 5.9)

(Figure 5.9 shows Bella's thinking)

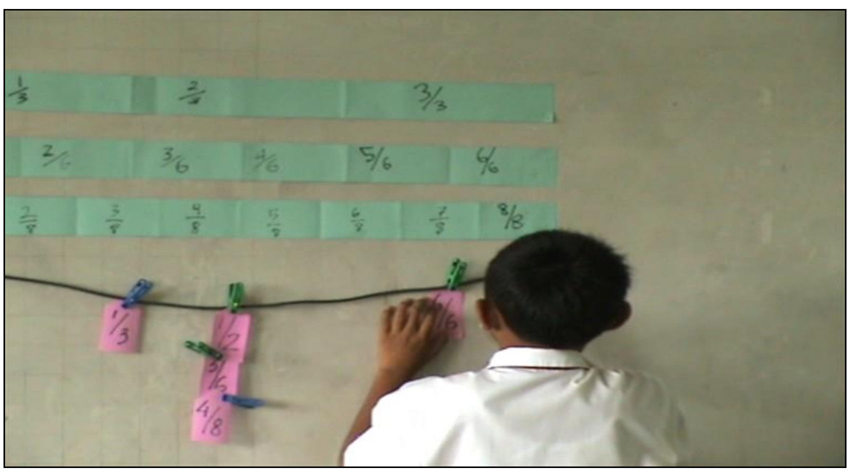
R: Was there anyone who could help Bella position the 3/6?

A2: Here, in front of the ½.

(Figure 5.10)

(Figure 5.10 shows Bella& Alya's agreement)

There was another episode in which students still struggled with the non unit fractions (I asked students to show 6/6.). Although 6/6 could be seen above the rope (from the symbolized paper folding), Syarif made a mistake in putting the mark for 6/6 on the rope. For him 6/6 was not the entire rope or was not on the edge of the rope. Instead, he positioned 6/6 on the rope in the same line as the starting 6/6 of the bar.



(Figure 5.11)

(In Figure 5.11 shows Syarif posited 6/6)

That confusion in positioning the fraction's mark on the paper bar folding and the rope happened because of a missing instruction that should have been given when they symbolized the paper folding. From the episode above, my conjecture is that of Bella's thinking of positioning 3/6 in the middle of the 3/6 square was encouraged by the writing of the symbol itself. 3/6 was written in the middle of the 3/6 square bar above the number line, not near the last line of the square. This also happened to

Syarif. My conjecture is that Syarif's thinking of positioning the 6/6 was inspired by the written symbol of 6/6 in the paper bar above the number line. 6/6 was written near the starting of 6/6 of the bar. Based on that, I believe that there was a missing guiding question that led the students to write the notation near the last line of each square. I realized that it was rather dangerous if students wrote the fraction's symbols on the middle of each paper folding square or near the starting line of the bar. If the instruction had guided the students to symbolize the fractions near the last line of each square, it would have decreased the students' confusion. For the future learning cycle of symbolizing the paper folding, it would be better if we just let the students write the fractions' symbol very close to the last line of each square. In the next part, we will describe the investigation of the common denominator through the contextual situation of the cake.

5.3 *The investigation of the common denominator should not be done through an abstract way of learning*

5.3.1 The rationale of the shopkeeper context

After exploring the relation among fractions through the number line activity, in the next part, we continued to the introduction of the common denominator of fractions that was required for the simple operation with fractions (e.g. addition of fractions) within a contextual situation and not using abstract ways of learning.

The students need to investigate the common denominator before they learn simple operation with fractions. In traditional ways of teaching, teachers usually teach strategies to find the common denominator of fractions in abstract or procedural ways. In other words, students are normally trained to have the procedure of finding the common denominator without understanding why they need to do it. In the observation that was conducted before the experiment (I observed another class when students were still in grade 3), the teacher taught students by letting them make a list of multiples of every denominator and then the least common multiple of it will be the common denominator.

e.g. Give a problem 1/3+1/4 = ?

Below the strategy of how to find the common denominator is indicated:

3 : 3, 6, 9, 12, 15, ...
4 : 4, 8, 12, 16

So the common denominator is 12.

At some point, I asked one student why he did it like that. He gave a simple answer; "because my teacher taught me in that way"

The shopkeeper context was one way to avoid an abstract way of learning. In this activity, the students will rationally find the common denominator by thinking about the given situation. The contextual situation helps students to avoid the procedural strategy and to understand the reason why they need to do it.

In a classroom experiment, the teacher introduced the contextual situation of making one measurement to help the shopkeeper serve the customer. This is how the story was going. The shopkeeper sold six different pieces of cake such as 1/3, 1/2, 2/3, ¼, ¾ and 1/6. Because the shop was very crowded, the shopkeeper needed the students to help him/her find one measurement tool to count for the fractional cake in order to serve the customers. If the shopkeeper used six measurement tools, he would need more time to serve the customer. We can say that this activity was related to the paper folding activity and the activity of symbolizing the paper folding, because we brought the students back to those activities first and afterwards students explored the common denominator through the given context. I expected that students would easily understand the introduction of the contextual situation by the teacher.

5.3.2 The research experiment of the investigation of the common denominator

In exploring the common denominator through the shopkeeper context, the students were considered successful in comprehending the concept. At the beginning, the students did not understand the story of the contextual situation though, but then they could understand it and the activity worked out well. The activity was done over two days. To explore the students' reasoning, the math congress about the activity of the common denominator was conducted two days after the discussion in a small group. In the math congress, all students were challenged to share their ideas. I found some surprising ideas and reasoning.

One of the examples was the idea Anas had when I asked him the common denominator of ½ and 1/5. He answered 10, and he reasoned that it came from the multiplication of 5 and 2. Then, Rheina came with a different reasoning, when asked about a paper bar that could be folded into a number of ways with three fractions: 2/6, ½ and 2/3 (I used the paper bar as the shopkeeper's measurement tool). She definitely answered 6 because it could be divided by 6, 2 and 3. Rheina's reasoning was used by all students as their strategy when they had to find a number that could be divided by all given denominators. The next challenge for the students was to give them more fractions and they had to find the common denominators of these fractions (1/2, 2/4, 3/6, 1/3, 5/6). Just in some seconds some students shouted and answered 12. Farrel made clear by reasoning that 12 could be divided by 2, 4, 6 and 3.

A good discussion among the students about the common denominator also occurred in the next episode, when they learned addition of fractions. After students solved the addition problem ½ + 1/3=5/6, the discussion was held among Rheina, the teacher and Firda, encoded as Rh, T and F2 respectively. There was a wonderfully unpredictable question from a student that did surprise me, and never had I thought about it before during the discussion.

Rh: Teacher, why did we use 6 instead of 12 as denominator?
T: Was there anyone who could respond to Rheina's question?
F2: I think six was enough.
T: If I use 12, what do you think?
F2: It was too much.
Rh: But I think 12 was also correct because the result of 5/6 and the result with denominator 12 will be same or equivalent
T: If for example Rheina uses 12, was that correct?
All students: That was correct.
T: If we used 12 as denominator, what fraction did you get?
Rh: I know...I know (raised her hands)…10/12.

The question from Rheina surprised and made me realize that the chosen denominator should be discussed among the students in the common denominator activity. Notably, I had not thought about that question in my Hypothetical Learning Trajectory. Sometimes students' thoughts develops further than what we assumed.

As explained, the students did not immediately understand the introduction of the given contextual situation. The difficulty of understanding the contextual situation at the beginning of the activity can be caused by two reasons: The students looked tired at that time and had difficulty to concentrate (after they just finished following the field games commemorating the Independence Day). Another reason was due to too much time used by the teacher for reminding the students about the idea of equivalent fractions. As a result, the students got confused in following the story. It could be seen from the transcript below when I, as the researcher (R), interviewed and guided two students, Faiz (F1) and Iko (I)

R: Do you understand your teacher's explanation?
F1: a little bit
R: Okay, I retell the story that there was a shopkeeper who uses six measurement tools to cut some pieces of cake that he sold. He sells some part of cake: 1/3, ½, 2/3, ¼, ¾ and 1/6 of cakes. Would you please help him to make one measurement tool that makes him quicker in doing his job?
F1& I: (silent)
R: try it by your folded paper bars.
F1: Use ½
R: Okay with this paper folding into 2, could we make a third?
F1: No, we could not, how about a third paper folding?
R: Could we make a half with that paper?
F1: Oh no no, we could not
R: Think of one measurement tool that could be used for six cakes?
F1:I try to think of 6. mmh 1/6, ½ then 1/3 was equal to 2/6, 2/3 was equal to 4/6
R: How if I want to buy ¼ of a cake and I use this 6 folded paper?
I: No, it could not be used
F1: I try 8..mhh I could make ½ mmh but I could not make 1/3
I: I try with 10 paper folding…mmhh I could make ½,..No, no I could not make ¼
Ah…I think with 12 paper folding
R: Why do you think like that? How it could be?
I: (while paid attention to his drawing of making 12 squares) ½..1/3…2/3..1/4…3/4…1/6..yes we could

The discussions above were held after the students did the paper folding activity and gave a fraction symbol to the paper bar by shading it or marking it to show every part of the cake. The shaded paper folding as a tool and the shopkeeper context helped Iko and Faiz to find one measurement tool for the shopkeeper (the common denominator). They understood the context, although they did trial and error first until they found the right solution for the problem. Iko's contribution surprised me, since in class Iko was a student who had difficulty in sharing his ideas. He was also very often losing his concentration in following the lesson.

It seems that we need extra time to guide the students until they get the right conclusion using their logic. Although Faiz and Iko found the common denominator after we gave some guidance, I still was not satisfied with this condition since I did not get their reasoning. They just used a trial and error strategy and they did not see the structure to reach the answer without the teacher's guidance. But in the math congress, students could share their reasoning about the common denominator activity.

From the discussion, interview and "taken as shared" among the students, we can conclude that the common denominator can also be introduced through a contextual situation that makes students avoid the technical ways and use their logical thinking strategy. The bar model help the students investigate the common denominator, not in abstract ways and this also explores the students' reasoning about it.

5.4 *The bar as a model for thinking and reasoning*

5.4.1 The Cake's context helps students avoid procedural ways in solving addition of fractions

After the activity of the common denominator, the learning process continued with the simple operation with fractions. The learning process continued with a discussion of the addition of fractions. In Indonesia, students learn the addition of fractions in grade 3. They learn it directly through abstract ways of learning. Although students had been introduced to the procedure of solving the addition of fractions, I observed that all students with whom we worked forgot their initial knowledge of solving addition of fractions' problem. This was indicated in one episode when the teacher asked the first question about addition of the fractions $\frac{1}{2}+\frac{1}{3}$. All students raised their hands and immediately shouted 2/5 together. In another section, the teacher wrote a question on the whiteboard $\frac{2}{3}+\frac{1}{4}$. Rheina doubtfully answered 3/8 without giving a description of her thinking. My hypothesis is that she used the cross multiplication's strategy $\frac{2}{3} \times \frac{1}{4}$. What can we learn from those two instances? The direct abstract way of learning did not make sense for students and it made them totally forget how to solve the problem. This was because they were trained to master a procedure without understanding it.

To avoid abstract ways of learning, students were taught by using the contextual situation. I developed a context of a cake, since I believe that the cake context was familiar to the students, because this was related to the previous context (the shopkeeper context to get the idea of common denominator). If the students needed to remember the common denominator, the teacher could still guide them to the one measurement tool in the shopkeeper context. Also, the shape of the cake that was introduced to students was a bar shape that would help students to think and reason.

The contextual situation for addition of fractions activity was about the customers who wanted to buy two different sizes of cake (fraction’s part). The students would help the shopkeeper find out the total amount that was bought by the customer and the price that should be paid for a whole cake was Rp. 12.000,-. We gave the price in the contextual situation since we realized that it would be strange if we asked students to add two different things, for example, the total of two mangoes + three candies. Another reason for giving the price was that the price was as an intermediate step to bring the students to add the two fractions of cake together. The price was also a way of helping students to find the common denominator.

5.4.2 The teacher’s guidance in the research experiment of addition of fractions

Before the teacher gave the addition of fractions’ problem, she guided the students by reminding them of the previous lesson of having one measurement tool if they had some fractions of cake. It did work. Anas and Iwan, henceforth indicated as A and I respectively, were the first group who shared their idea of solving the addition of fractions at that time. They came in front of the class and explained their strategy of solving ½ +1/3

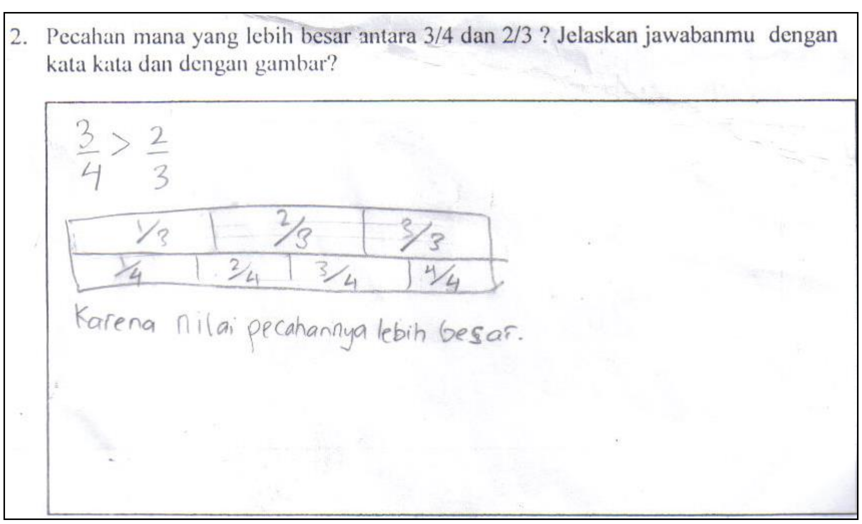
I: ½+1/3, first 1/2, we looked for the denominator 6
R: Why do you use denominator 6?
A: Because 6 could be divided by 2 and 3. Then the equal fraction of ½ was 3/6 and the equal fraction of 1/3 was 2/6 so ½+1/3=3/6+2/6 was equal to 5/6

Almost all students were able to solve the question above. Therefore, it can be inferred that the addition of the unit fractions could be well solved by the students. Afterwards we tried out the addition with the non unit fractions. Here is how the story went. A customer bought 2/3 chocolate cake and ¼ vanilla cake and the price of each cakes was Rp. 12.000,-. One of the students (Rheina) responded and shared her thinking with me. Notably, Rheina had a great development of thinking and learning while following the sequenced activities (based on the teacher’s information, before I came to the class, Rheina was a passive student and she was not really joining in her class). First of all, she could easily find the twelve as the denominator. Then she told me that one fourth was equal to 3/12. I told her to continue with the other fraction (2/3). She was silent, but after some seconds she drew a bar and divided it into 12 to help her find the equivalent fraction of 2/3. I brought her to the unit fraction first and find the equivalence of it.

I thought that students were more familiar with the unit fractions instead of the non unit fractions. I asked "How many squares are there if you look for 1/3?" She answered four squares and had 4/12 as the equivalent fraction, while she also shaded four squares of the twelve squares. Then I continued my question: "so, how about 2/3?" She still played with her drawing and she drew the 2/3 above the 12 squares in a bar. By discussion with Maria, Rheina found 8/12 as the equivalent fraction of 2/3. Then she got 11/12 as the result. This drawing of the bar was not only done by Rheina, but almost all groups went back to it when they struggled with finding the equivalence of fractions and the price of the total cake. For example when Rheina explained how she got the price of adding 1/6+1/4= 2/12 + 3/12 = 5/12. She said that because we had 5/12 (she drew 12 squares in a bar) then she continued: "Because there are 12 squares, one square is Rp.1000,-, so 5 squares will be Rp.5000, - ."

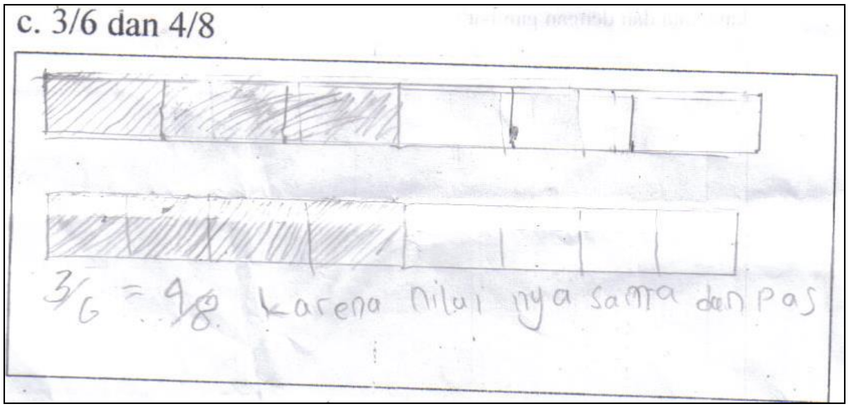
The teacher also used the bar to encounter different strategies and thinking of the students (when different answers appeared on the whiteboard). It was easier to get an agreement between students by drawing the bar. For example in the same problem as Rheina above, Hakim had a different answer for the price of the cake. Rheina got Rp. 5000,- for the total cake 5/12 and Hakim got Rp. 10.000,- for the total cake 10/24. The teacher guided the students by drawing a bar consisting of 12 squares and a bar of 24 squares. The misunderstanding among the students could be solved using the bar drawing. That is why it was important to have the model (for example: bar), because students could use the model to prove their answer or solution

Double bar drawing was used by students to support their reasoning in comparing two fractions. It could be seen from the student's answer in the given homework below:


2. Pecahan mana yang lebih besar antara 3/4 dan 2/3 ? Jelaskan jawabanmu dengan kata kata dan dengan gambar?

(Figure 5.12)

(Figure 5.12 is Fella's homework. The translation of the problem: *Which fraction is bigger? Explain your answer with words and with a drawing?*
Answer: *¾ > 2/3 because the value of the fraction is bigger*)



(Figure 5.13)

(Figure 5.13 is Hakim's homework. The answer's translation: *3/6=4/8 because the value is equal and exactly fit*)

There was a missing instruction in the addition of fractions activity. At the beginning, the addition's problem was given; the students did not really care about the price that should be found. This happened because the first problem that was written by the teacher in front of the class was just asking to add the two fractions rather than the total price. So most students paid attention to add the fractions first, and then after the teacher reminded them about the price, they found the price of those total fractions.

. When I observed the whole class, I found that students needed more time to find the equivalence of the non unit fractions. The difficulty of the students in finding the equivalence of the non unit fractions is described in the next part.

5.5 The difficulty of finding the equivalence of the non unit fractions by fourth graders

In the discussion of the addition of fractions, most students struggled with finding the equivalence of fractions after they found the common denominator, especially for the non unit fractions. For example, it was shown in the episode of discussion between me henceforth indicated as R, and some students: Alya, Faiz, Iko and Maria, encoded as A2, F1, I and M respectively. The problem that they faced was ¼ + 2/3. They struggled with finding the equivalent fractions of 2/3 with 12 as the denominator. What I did was guiding those students to the unit fractions first, then going to the non unit fractions, since they were more successful at mastering the unit fractions.

R: if you have 12 squares and how many squares it should be for 2/3?
A2, F1, I, M: (Silent)
R: I have 12 squares (while I made a bar and divide the bar into 12), so, how many squares do you have for 2/3?
(A2, F1, I, M tried to shade the bar and looked for the equivalence of fraction of 2/3)
R: Okay, let's back to the problem: ¼ +2/3. Do you know the equivalence of fraction of ¼?
M, A2, F1, I: 3/12 (then they wrote ¼+2/3=3/12+… on their worksheet)
R: Okay, now how about the 2/3?
A2: 4/6
R: Yes, but we have the 12 squares, looking for it again.
A2, F1, I, M: (silent)
R: Okay, let's look for the 1/3 first in the 12 squares?
A2: four
R: okay four (while I made shading on the 4 squares of the bar), See if you have 4 squares for 1/3, How about the 2/3? (While I pointed to the shaded squares)
M: eight squares.
R: So, what was the fraction?
M: 8/12 (then they write ¼+2/3=3/12+8/12 on their worksheet)
R: Everyone understood?
A2, F1, I, M: (nodding their head)
R: Could anyone tell the result?
A2, M: 11/12

From the description in one episode above, we could see that those students struggled with finding the equivalence of fractions, especially for the non unit fractions. The activities which focused on finding the equivalence of fractions took the most part in the sequence of activities. There were four meetings working with the equivalence of fractions with two contextual situations (the paper folding activity and the rubber bands activity) that were given before students learned the simple operation with fractions. There might be a problem in the paper folding and rubber bands activities. Therefore, in the following, we describe the paper folding activity and the rubber bands activity.

5.5.1 The paper folding activity for exploring the equivalence of fractions

The learning goal of the paper folding activity was that students would explore the equivalence of fractions within the activity. In the activity of the paper folding, the students were given some paper bars to be divided into 2,4,3,6,8,5,7. I expected that the students would realize and find the equivalence of fractions while they folded the paper into certain numbers. For instance, when the students folded the paper bar into two, they would make ½ and then each half woud be folded into two again, so they realized that two of one fourth was ½. Then students could conclude that ½ was equal to 2/4. The context of dividing a cake was the contextual situation for the paper folding activity. The paper bar was as a model for the cake. To observe and explore the idea of equivalence of fractions in this activity, we had a math congress in which we gave some guidance.

Most students were more focused on how to divide the paper bar. They had not found the equivalence of fractions idea when they did the paper folding. This was caused by the worksheet that was given to students, which asked them to describe the way they folded the paper bar. We missed the guidance of finding the equivalence of fractions in the worksheet. Realizing my mistake in making the worksheet and seeing that the students didn't pay attention to the fractions within the paper folding activity, I interrupted some groups and guided them to the idea of the equivalence of fractions. There was a group of students (Anas(A1), Bella (B) and Nouval(N)) who easily folded the paper into two and three. Afterwards I (R) guided them when they did the paper folding into four.

R: How much part was it? (While pointing to one part of four paper folding)

A1: one of….

B: One fourth

R: Is there anyone who can tell me the half of the paper?

(Figure 5.14)

Figure 5.14 shows Nouval pointed to the paper bar. Afterwards, I guide this group by repeating asking the one fourth and asking the half of it.

R: What you could conclude?

B: mmmhhh a half mmmhh was equal to two fourth

A1: three over two (look confuse)

B: See there were four squares; the middle of it was two squares so 2/4 was equal to ½.

T: What do you think Nouval?
N: yeah that's correct (while counted one by one the square and recounted the two of it)
T: How about you Anas?
A1: I feel confuse.

Anas' confusion made me decide to have a math congress on the same day because I believed there were more students, like Anas who still struggled with the meaning of fraction and the idea of equivalence of fractions through the paper folding activity. My belief was proved when the math congress was held. I asked Nouval to share their strategy of dividing the paper into 8 with the whole class:

N: First I fold the paper into two, and then fold again into two
R: What we got if we first folding the paper into two?
S(students): a half
R; okay then you continue fold it into two again, see we got four parts
N: Then we fold into two again
S: we got eight.
R: Where was the half?
S: four parts
R: So what was the fraction?
H (Himmah): 4/4
R: Was the ½ equal to 4/4?
Some students: yeah

Afterwards, I observed some students who still struggled with the equivalence of ½. What I did afterwards was to show a paper bar that had been folded into two, and under that paper, I also showed a paper bar that was folded into 8. I continued to ask the students what the equivalence fraction of ½ was. Some students, including Himmah and Thoni still answered 4/4. I reminded them that the total fold of the paper was 8 and then Fella corrected her friends by answering that ½ was equal to 4/8. Although some students in the class had agreed that ½ was equal to 4/8, Himmah and Thoni were still insisting on their answer that ½ was equal to 4/4. Then, I asked all students, "What does 4/4 equal to?" Rheina and her group answered "one". Based on Rheina's answer, there was an agreement between students that ½ was equal to 4/8. That struggling situation happened again when I showed ½ papers folding above the six paper folding. Rama thought that ½ was equal to 3/3 until agreement was attained that ½ was equal to 3/6.

What happened in the class contradicts the result of the pre-test. In the pre-test, most students seemed to understand the meaning of fractions as a part of a whole. The result could be contradictive since in the pre-test, students described the fractions of the shaded pictures, but in the episode above, students saw the paper folding without any shading. Students could not work out the idea of equivalence of fractions when they folded the paper. There are some factors that have caused this problem. My analysis is that the given instructions for the activity of the paper folding were not focused enough on exploring the equivalence of fractions, and the context was not strong enough to achieve the idea of equivalence of fractions. In addition, the worksheet itself made students focus only on exploring the strategy of how to fold the paper. If we see this from the students' point of view, they have learned the basic fractions in grade 3. The students were used to learning the formal level of fractions in advance. Because of the limited time and the students' struggling with indicating the fractions, we just explored the equivalence fractions of "a half" in 8 and 6 papers folding in the math congress.

5.6 The refinement of exploring the equivalence of fractions through the rubber bands activity

5.6.1 The rationale of the rubber band activity

To refine the students' understanding of the equivalence of fractions, a new contextual situation was introduced to the students. The context was about the Indonesian Independence Day's game (cone cap games) context, with the rubber bands as the new model. We had done the try out in the rubber bands activity with some students in grade 3 before the research experiments were held. This activity gave a wonderful result of finding the equivalence of fractions in the try-out phase. The students could explore the equivalence of fractions through the rubber bands activity. That was why we put the rubber bands activity in our sequence of activities in the research experiment.

Why do we use the string rubber bands as the model here? Because the string rubber bands can be stretched so that it can be used to find the equivalence of fractions logically. Mathematically, the idea of the rubber bands activity was that students would come to the process of finding the fair area lines to hang the cap for the games (the equivalence of fractions) by stretching the rubber bands. The string rubber bands can be used as rope to hang the cap for the game. The students were asked to make fair line divisions in the string rubber bands based on the number of participants in the game. For instance, students want to divide a line into 4 since the number of participant is 4, they can use and stretch 4 string rubber bands to have a fair area line. If the four string rubber bands are not stretched enough, they can use 8 string rubber bands to be divided into 4. Furthermore, if 8 string rubber bands are still not enough, I assume students will use 12 string rubber bands, since 12 can be divided into 4.

I gave an example of the logical thinking that could be explored through this activity. By doing the activity like the example above, students will find the equivalence of fraction 1/4=2/8=3/12… The rubber bands activity also helps the students to solve the unsolved problem in the paper folding activity, which is to make a division into odd numbers (5 and 7). With the string rubber bands, students can divide something into all numbers, even into odd numbers.

In fact, it might be better to do this activity in a field, because it has been found to be more suitable with the context of the cone cap games, but at that time, we could not do so because of the school's rules. Therefore, the activity was held in the classroom. Although the activity took place in the classroom, the story of the cone cap games was still used and introduced to the students because the model (the big rubber bands) that was used still corresponded to the context of the cone cap games. But because of the limiting condition, we used tables and whiteboards so we could measure fairly. The overview of the activity is shown below:

(Figure 5.15)

(Figure 5.16)

5.6.2 The finding of the rubber bands activity

I observed that all students understood the context of measuring the table with the rubber bands. All students understood that they needed to add more rubber bands when the stretched rubber bands were not long enough (the number of the rubber bands should be determined by the students, thus it could be divided based on the task given). There were two different logical strategies emerging from the students through this activity. First, there were some students who used their logical thinking strategy of having the equivalence of fraction through the rubber bands activity (dividing a length). They divided a length into the number on the task given. Then the fractions would be the result of the division of the rubber bands/the total rubber bands, e.g. the task was dividing a length into two. The students used two rubber bands and they had ½, but it was not enough, so they added two more rubber bands (4 rubber bands) and divided into 2, the fraction they had was 2/4, etc.

The second strategy of the students came from the teacher's idea that students found the equivalence of fractions from how many times they added the rubber bands over the total number of the rubber bands. For example, it was shown in the conversation between the teacher (T) and Rheina(Rh) in front of the class. Rheina shared her strategy in dividing three tables into four parts with all of her friends:

Rh: First, we want to divide three tables into four parts. It was still not enough and we added four more string rubber bands. But it still not enough. Then I added four more rubber bands.

T: How many rubber bands do you have?

Rh: 12 rubber bands

T: How many times do you add the rubber bands?

Rh: three times

T: So, what fractions do you have? Please write it on the white board (Rheina wrote ¼)

T: How do you get ¼? Where was it come from?

Rh: we have four rubber bands

T: ok, how about the "one", where was it come from?

Rh: We just measure it once so I got ¼ (she wrote ¼ on the whiteboard). Then I measured for the second time and the rubber bands were eight still not enough (while she said this sentence, she wrote ¼=2/8). And then the third measured was enough with 12 rubbers, so 3/12.

The teacher's guidance in the conversation above was too procedural and made the students under her guidance forgot the contextual situation that was introduced beforehand. In my opinion, the teacher forgot the contextual situation that was given.

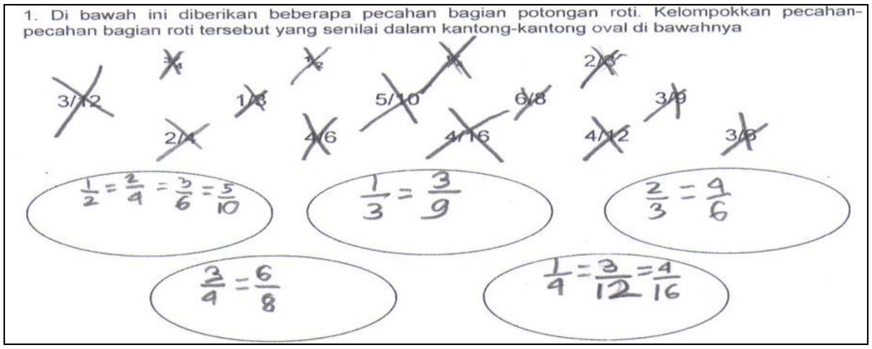
I realize there was something missing during the activities, as I did not guide the students to explore the equivalence of the non unit fractions. For instance, we could ask students to show 3/4 of a whiteboard using the string rubber bands. If we ask students such a question, I assume that they would use four string rubber bands and show ¾ as the three rubber bands. There were not enough string rubber bands to be stretched, so I expected that they would use 8 string rubber bands and found 6/8 as the equivalence of ¾. What happened in the class was that the instruction brought the students to start with the unit fraction. They followed the teacher's instruction that counted the time of measuring and the number of the rubber bands used. For example, the first time they measure using two rubber bands, they will get ½, etc.

On the whiteboard and the given worksheet, the students always started by writing the unit fractions and then continued by finding the equivalence of the unit fractions. In other words, the rubber bands activity in the research experiment forced students to start with the unit fractions and then find the equivalence of the unit fractions. What I was worried about regarding student's struggle on the next activity truly happened. I found that some students still struggled with finding the equivalence of non unit fractions when they were interviewed by me and solving the post-test given. To face this problem, more instructions on how to look for the equivalence of the non unit fractions, like the example above, are needed in the rubber bands activity to find the equivalence of non unit fractions. Besides, the bar drawing was also a suitable model for thinking and helped students to find the equivalence of non unit fractions.

5.7 *The Post assessment of students' learning fractions*

The post-test was given to all students in order to measure students' understanding after they followed the learning activities. There were four problems with sub problems in it. The problems were related to the sequence of activities. There was no connection between the pre-test that was already given before the activities hold with the post-test. The post-test was given in order to measure the understanding of the students after following the learning activities. If they were successful in solving the task in the post-test, their work indicates an understanding in progress. We shall not only describe the answers of the students, but we will also consider the students' reasoning in the post-test.

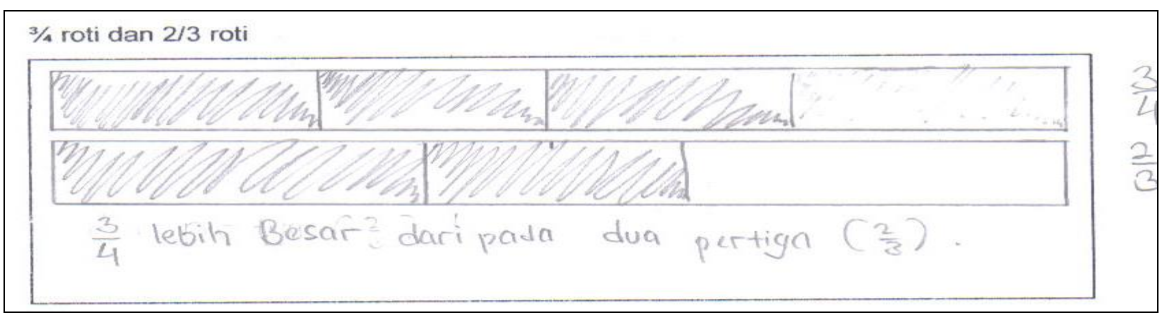
The first problem was about grouping the equal fractions. There were some fractions in random order and what the students had to do was to make some groups of equivalent fractions in the given oval spaces. Most students could group correctly to unit fractions such as 1/2, 1/3, ¼ and they made some small mistakes in the non unit fractions. But there were also some students who could group all fractions correctly. One of them was Faiz. Below was shown Faiz's work :



(Figure 5.17)

(The translation of problem figure 5.17: *Following were some cake fractions. Group those cake fractions that correspond to the given oval*)

The second problem was about comparing fractions of the cake. In this problem, students were not only asked to compare the two cake fractions, but they also had to state their reasons to support their answers. Different reasons come from the students. The reasons that they gave, were related to the sequence of activities that they followed. This indicates that the students could make sense of the activity. Most students made double bars to support their answers. For these students, I assume that they remembered the paper folding activity. One of the examples was the work of Iwan. He made a double bar, then shaded the fractional part of each bar and decided which fraction was bigger. It is shown in figure 5.18 below:

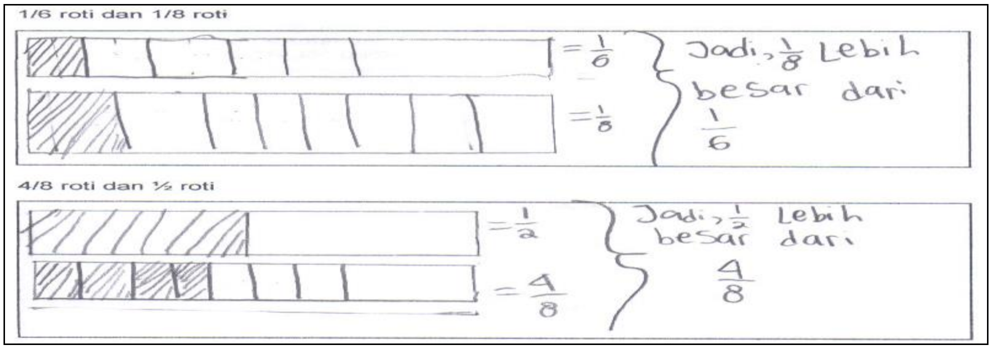


(Figure 5.18)

(The translation of figure 5.18: *3/4 was bigger than 2/3*).

Iwan used the double bar to support his answer. He used the bar model for representing the fractions.

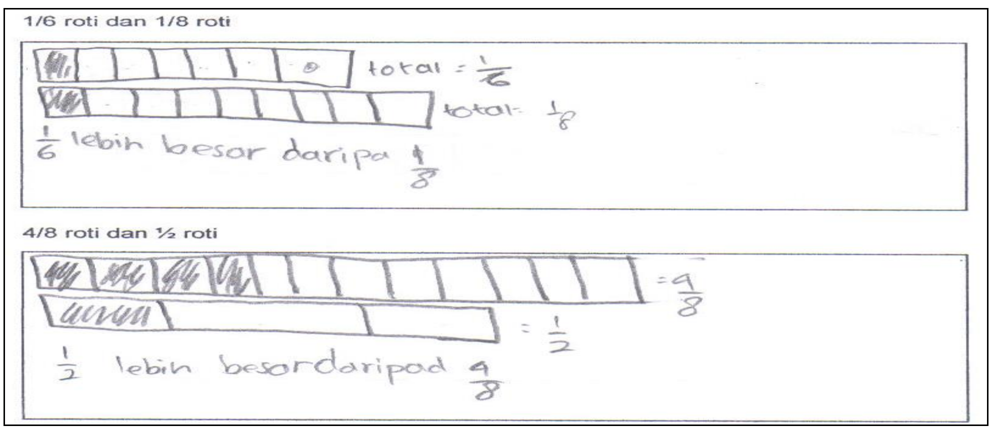
However, there were also some students who drew the double bar and made mistakes in deciding which fraction was bigger or smaller. This happened because those students did not make a fair division or drew different sized bars indicating unequal units.



(Figure 5.19)

(Figure 5.19 shows Danti's work. The translation: *So, 1/8 was bigger than 1/6. So ½ was bigger than 4/8*)

From Danti's answer above, we could see that Danti really believed her drawing instead of her reasoning, and she had answered the questions based on her drawing. She used the bars drawing as a tool instead of reasoning

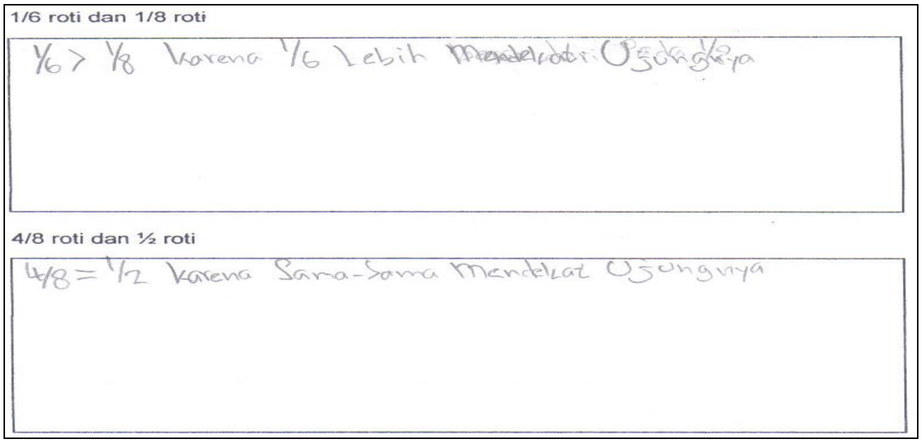


(Figure 5.20)

(Figure 5.20 shows Alifia's work. The translation: *1/6 was bigger than 1/8. ½ was bigger than 4/8*)

Alifia's work shows that she does not understand how to use two equal bar units to be divided fairly and show the fractions. She used two different bar units of the bar and showed the fractions in each bar and compared it.

Other different reasoning that came up about this problem was related to the number line activity. Beside those two different ways of reasoning, the rest of the students did not give their reasoning. They just gave their answer and did unequal division (making a double bar with the first bar divided horizontally and the second bar divided vertically)

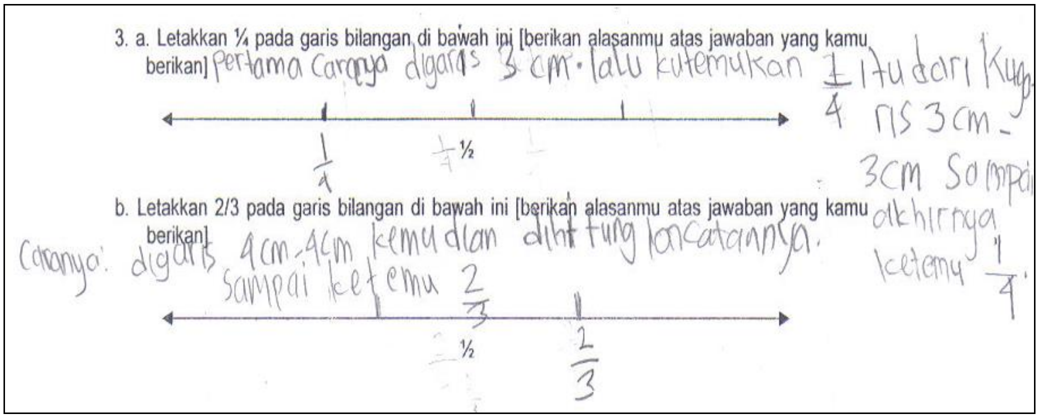


(Figure 5.21)

(The translation of Himmah's work in figure 5.21: *1/6>1/8 because 1/6 was nearer to the border. 4/8=1/2 because those were same nearer the border*)

The number line activity made sense for Himmah. He wrote that 1/6 was bigger than 1/8, because the position of 1/6 was closer to the edge. In my opinion, what Himmah means by the edge here was the last edge point because he only reasoned about it in one episode of the number line activity when we asked the comparing fractions of 5/6 and ½. He answered that 5/6 was bigger, because it was further and goes near the last corner of the rope.

The third problem was related to the number line activity in which we give two sub problems of the number line. In the number line, we put ½ as the anchor point, since I believed that fraction was familiar to the students. In this problem, 75% of the students could solve the problems correctly, 20% of the students made one mistake from the two sub problems and the rest could not answer the questions. Alya was able to position ¼ and 2/3 in the number line. She also described her strategy to place the fractions. Below, we show Alya's work in figure 5.22.



(Figure 5.22)

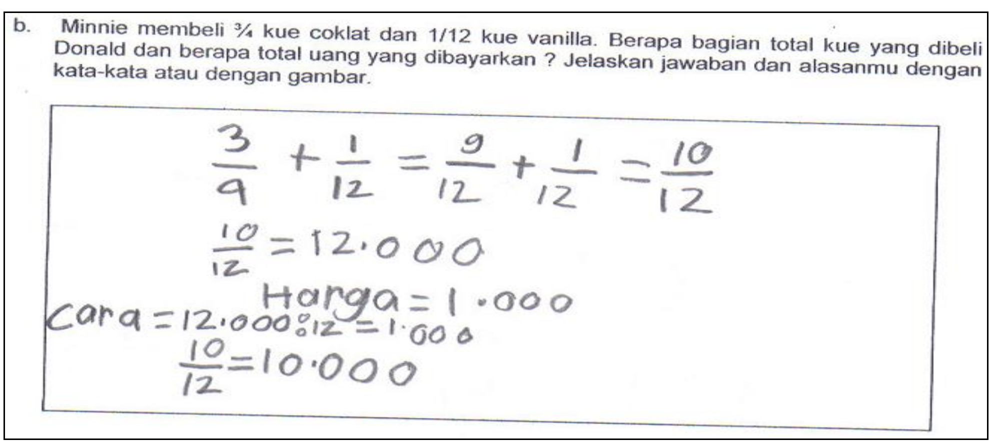
(The translation of Alya's strategy of figure 5.22:

3.a. First, I measure 3cm then I found ¼ from measuring 3cm-3cm, at last I found ¼

b. 4cm-4cm was measured then the step and find 2/3)

With her work, Alya wanted to show the exact point for ¼ and 2/3 by measuring it.

The fourth problem was related to the simple operation of fraction (addition of fractions). The cake's contextual problem was similar to the context of the addition of fractions' activity. Students were asked to find the total fractions of cake and the total price of the cake that should be paid by the customer. From the students' answer, 50 % of the students succeeded in solving the problem, while the rest could not solve the problem properly. The students who could answer the problems properly and correctly used the double bar to find the equal fractions after they found the common denominator. However there were some students who did not use the double bar to help them. Below was an example of work in which the student did not use the double bar to help him.



(Figure 5.23)

(Translation of the problem in figure 5.23 that shows Faiz's work : *Minnie bought ¾ chocolate cake and 1/12 vanilla cake. How much cake was bought by Minnie and how much money should she pay? Explain your answer with words or with picture*) Note that, in the main question, it was explained that the price of every cake was Rp. 12.000,-

Faiz could find the equivalence of each fraction and arrived at the total fractions. The price could also be calculated by Faiz. The 50% of the students who could not answer the question properly just made mistakes in deciding the total price, although the addition of fractions could be answered correctly. Another mistake committed by those students was just to compare fractions with the double bar that they made. To my analysis, those students did not have enough time to complete their work.

5.7.1 The Analysis of the post-test

From the result of the post-test, in the first problem, it is clear that some students could group equivalence of fractions if they started with the unit ones. However, I assume that most students struggle in finding the equivalence of fractions of the non unit fractions.

The second problem of the post-test that is about the relation among fractions, such as comparing fractions, could be solved by the students and was followed by their reasoning to support the answer. The reasoning that they gave was related to the activities that they followed, such as the paper folding activity and the number line activity. The students, who reasoned with the activity that they already had followed, had indeed learned to use the model in the activity for their reasoning. Most students drew a double bar to reason about comparing the fractions, for example by drawing the double bar with a fair division. However, there were two students who used the bar not as a model for reasoning about fractions, but rather as a tool. They really believed their bar drawing notwithstanding no fair division. As well as using the bar drawing, there was also one student who used the number line as a model to support his reasoning. He reasoned with the number line in that the bigger the fraction, the more to the right the position was. Most students were also able to position two fractions in the number line with ½ as the anchor point. Based on the explanation above, the bar model and the number line model were both comprehended by the students with whom we worked to help their reasoning.

Half of the students could solve these problems of addition of fractions. However, the rest still found it difficult to solve them. Most still found it difficult to come up with the equivalence of the non unit fractions. My analysis was that this might have been caused by misguidance about the non unit fractions that should be given within the activity of the equivalence fractions.

Besides giving a written assessment, I interviewed some students to identify their impression of the given activity and to test them directly through the fraction card games. An interview with one of the students will be described in the following part.

5.8 Rheina's experience and reasoning of doing the activity

All students with whom we worked had followed the sequence of activities completely. As the researcher, henceforth indicated as R, I was really curious to hear their opinion about their experience during the learning activity. One of the ways to know their impression was by interviewing some students after the completion of the sequence of activities. I did the interview with Rheina, henceforth indicated as Rh. She told about her difficulties in solving the given post test as well as explaining her experience in studying fractions when she was in grade three and after following the research experiment.

R: Rheina, how was the post test you had? Was there anything difficult?

Rh: a little bit

R: Could you explain it a bit more?

Rh: In the addition of fraction with unequal denominator, I didn't understand a little bit at the beginning, but after Ust. Lyna (R) had explained it, I could understood it.

R: Which one was it?

Rh: Like this, Find the number above (she meant the numerator after she found the common denominator, while showing problem 4a, problem of addition 1/6+3/4)

R: I see, was that related to the equivalence of fractions?

Rh: Yes.

R: okay Rheina, before today, we had learned fraction through the activity which was given, have you ever learned fractions by this way of learning before?

Rh: Never. But I think I had ever been taught about fraction, we just looked at the white board, paper and book without playing like what you gave us. So, we go bored.

R: Then, what was the difference between the last time you learnt, which you just looked at the whiteboard, paper and book, and what we did?

Rh: Now, I feel I have spirit to study and it's nice because we could play and learn.

The conversation above shows Rheina's impression and feeling about doing the sequence of activities. To prove her saying and see whether or not the given sequence activities made sense for her as they were, I continued the interview and gave some questions about fractions using fraction cards with several fractions in it and asked her to compare those fractions, and subsequently gave some problems in the card about the addition of fractions.

R: I hope you could answer as quickly as possible. If we have one third and one fourth, which one was bigger?
Rh: one third
R: How could that be?
Rh: I remember from the picture (she meant the bar picture that she made)
R: Okay, or probably, do you remember the activity we did?
Rh: Yes of course, the part of paper folding of 1/3 was bigger than the part of paper folding of ¼.
R: How about this? (I showed two cards with 2/5 and 2/6)
Rh: 2/5
R: Why was it Rheina?
Rh: the same as before, the part of the paper folding of 2/5 was bigger than the part of the paper folding of 2/6.
R: How about this? (I show two cards with ½ and 3/6)

(Figure 5.24)

Rh: Equivalent
R: Why could it be equivalent?
Rh: Because the shaded picture of ½ and 3/6 were equal
R: How could the shaded be equal? What were you doing with your finger? (Rheina played her finger, indicating that she might count something).
Rh: I am dividing
R: What were you dividing?
Rh: six could be divided by two because if we want to know the equivalence of fractions, we could see from the denominator. If one of the denominators could be divided by another denominator, those fractions were equivalent.
R: Wait, if we have ½ and 3/8, this could also be divided. Was it equivalent?
Rh: (look puzzling) mmh wait in rubber bands activity, if we have ½, it means once time measuring, and if two times measuring means 2/4 and if we measure three times, it will be 3/6, thus equal.
R: I see, you remembered the activity of the rubber bands. Okay, I have the next card (2/3+1/4)
(Rheina wrote the problem in her paper and she started with her fingers, and wrote again, then she kept silent for a moment to think, after that counted again with her fingers)

R: Could you tell me what you were doing and thinking from the first time you see this problem?

Rh: First, I found a number that could be divided into 3 and 4, and then I found 12. Then 2/3 was equivalent to with which fraction that the denominator was 12, then I found 8 so 8/12.

(Note that for ¼, Rheina wrote that the equivalence of it was 3/12)

R: Where does the "8" comes from?

Rh: if 12 divided by 3 was 4 (while counted with her fingers), so I think the 2 should be multiplied by 4 and I got 8. So the result will be 11/12 (while counted with her fingers)

Rheina seems to be able to make sense of the paper folding activity in that she used the bar model of this activity to support her reasoning of comparing fractions. She always drew pictures of the bar model to represent her thinking and reasoning. Rheina also remembered the rubber bands activity when I asked her to compare the fractions of 3/6 and ½. The rubber bands activity was truly compelling for her. Also, she used the bar model to help her find and reason about the equivalence of fractions when she solved the addition of fractions' problem.

Most of the time, Rheina played with her fingers. In my opinion, this was done in order to calculate operations of division, addition and multiplication. Calculation by fingers was not only done by Rheina, but also some of the other students in completing their arithmetical calculation.

5.9 Reasoning for Indonesian students

It turned out to be quite difficult to explore students' reasoning in some of the first activities, since they were not accustomed to giving their reasoning. When I observed and interviewed them, I think that most students felt afraid of making mistakes in answering the given questions and some of them also felt ashamed. As seen in the paper folding activity, students were given a worksheet to describe their strategy and explain their reasoning about their findings. On the basis of my observation, they used most of their time to discuss how to formulate a good sentence describing their strategy in the given worksheet. They were frightened and worried to make incorrect answers/sentences. In another episode of a math congress, the students looked ashamed in giving their answers and reasoning. This could all be caused by the students for not being able to give their reasoning. More important for them was that they could come up with the final answer of the question without giving their reasoning, i.e., how and why they could reach their answer.

In my opinion, a little force seems to be required for the students to let them get used to giving their reasoning. That was why, in the adjacent activity after symbolizing the paper folding, the teacher asked questions to the students and let them be responsible for giving their reasons to their answers. The "How" and "Why" questions were asked after each activity to train the students with whom we worked to describe and explain more about their strategies and why they came to their strategy. This will train the students, especially Indonesian students, as they are not used to sharing ideas with others, to be able to give their reasoning. In order to encourage the students to be active learners and share ideas, the teacher and I gave motivating stars to students whenever they could share their ideas and be active participants in the discussion. Active participants' here refers to the students who could ask questions and criticize their friends' answers and reasoning. The stars could function as awards for the students; the more stars they got, the more proud they were. Giving stars was one of many ways that really worked to make the students active learners in sharing ideas and comment on what their friends did in the class.

5.10 The teacher's experience and comment

Besides exploring the students' impression about their experience, I also interviewed the teacher to know her impression about the teaching in the research experiment using Realistic Mathematics Education. In the following part, we will describe an interview session with the teacher after the teaching experiment.

After all the sequence activities were done and after the post test was hold, I, henceforth indicated as R interviewed the teacher (T) to explore her impressions of experience in contributing in teaching in the research experiment.

R: Would you please tell me your experience in teaching mathematics before you contributed in the teaching in the research experiment?

T: Beforehand, I never taught fourth grade students, I taught the third grade students.

R: How did you teach the student?

T: In teaching the students, what I did was based on the book and I feel that I gave directly to the abstract learning level, just following the book from the library. So in my opinion, what I gave was not for the students' world and I feel that the students were forced to follow my instruction. At least I brought them to the semi abstract way of learning and never used concrete examples. The students were also always afraid of sharing their ideas and reasoning because they were not used to it. Because of the limited time, I also found it difficult to use media and it was easier for me if I brought the students directly to the abstract way of learning to save time.

R: Okay. In the research experiment, you used the RME method in teaching through contextual situations. What do you think about that, I mean your experience in doing that? What do you feel?

T: Before I followed the research experiment, I feel that my students just made a note, saw me in front of the class and it was uncommon for me to use model or media. Then, what I saw in some meetings in the research experiment, I felt that students did not have to make notes and were not forced to see the whiteboard and they were moving. It was suitable with the characteristics of the students with whom we worked, since they always like to move. Beside that, it was good for them to study in a group to use models. But I feel not too satisfied with the condition in the research experiment, since I need to repeat and repeat again and need more time than usual, but I could see the happiness of the students when they learned. I was so surprised and just realized that actually the students have great development of thinking and they could be asked to think further than I expected of them. So there were some questions that enlarge their thinking and I just was a mediator without always giving them the knowledge directly. I just felt it yesterday when I taught using RME's method.

R: Could you approximate what percentage of students could understand and had a good development of thinking?

T: I saw about 60% of students, when I asked them to talk, they understood what I mean, but in writing, they felt confused again. At least they were quicker in responding me. So in my opinion, it needs more repetition and more habituation to them because they were usually let to think themselves for a long time and then wrote their thinking. They were not used to hold a conversation while writing.

From the conversation above, we could see that Mrs Evy as the teacher felt that there were great development in the students' thinking after they followed the RME learning method. We taught the students to be responsible with their answer by giving their reasoning and in the research experiment, they could think far beyond the teacher's expectation. Another interesting thing in the conversation above was about the difficulty of the students in explaining their thinking through writing. I agree with the teacher that the students understood if the teacher and I talked to them and guided them. They also understood what we were talking about and they could solve the given problem. But if we gave another of the same type of questions, sometimes they forgot and were confused about how to write their answer. It was probably because those types of students were fast thinkers, and it makes their writing insufficient. This type of students needs to be reminded, if not, they would forget again the strategy that they already had. I agree with the teacher that it needs more repetition and more habituation in this way of learning.

5.11 Conclusions

Upon the pre-test completion and deciding that the students' knowledge was sufficient to follow the sequence of activities, I explored the students' learning of relations among fractions such as equivalence and comparing fractions and simple operations with for instance addition of fractions that would have them move beyond learning the procedures without understanding per se. The aim of this design research was to develop students' understanding on fractions, the relation among fractions such as the equivalence of fractions and comparing fractions, as well as the simple addition of fractions

The activities in our Hypothetical Learning Trajectory constituted a sequence of activities from the equivalence of fractions as the basic idea and then continued to explore the common denominator and the addition of fractions. Operations on fraction should be delayed until the concept of fractions and the ideas of the order and equivalence of fractions firmly established [Bezuk & Cramer, 1989]. There were six main activities in the research experiment followed by a math congress, involving the paper folding activity, the symbolizing of the paper folding activity, the number line activity, the rubber bands activity, the shopkeeper context activity (exploring the common denominator) and the cake context activity (addition of fractions). Some of the activities were found to work out well as expected. The math congress of the symbolizing activity combined with the number line activity and the exploration of the common denominator through the shopkeeper's context worked well.

Through the number line activity, this could explore the students' understanding and reasoning of comparing fractions, in order for the students to keep away from abstract learning, for instance by doing the cross multiplication to decide which fraction was bigger than the other in its counterpart. In the number line activity, students shared their ideas and reasoning in comparing fractions, etc. The reasoning was related to the characteristics of the number line itself. The number line model was an abstraction of the bar model. The number line model was a suitable model to go further with fraction operations. The number line model embeds fractions in the set of natural numbers and facilitates the application of the natural number knowledge in the domain of fractions [cf. Menne, 2001 in Keijzer, 2003]. The meaning of a fraction was also discussed among students in the math congress of the symbolizing activity and the number line activity. In the post-test, there was a student who used and drew the number line to support his answer and reasoning of comparing fractions. It means that the number line model made sense for that student. Most students were able to posit fractions in the number line with ½ as the anchor point in the post test and they came up with their reasoning related to the number line, until there was a “taken as shared” reasoning of comparing fractions within the activity, in that the bigger the fraction is, the more to the right its position should be. Therefore, it is obvious to say that this activity was a rich activity.

The exploration of the common denominator through the shopkeeper context was also one of the successful activities. The shopkeeper context has an important role to bring the students to imagine the situation of the shopkeeper that mathematically develops the idea of the common denominator. This context is relevant in an everyday life situation, because students are familiar with the shopkeepers in their daily life. The students were successful in doing the investigation of the common denominator in a way that escaped an abstract strategy in learning. Students found the common denominator of some fractions per se, described the strategy of how they had arrived at that common denominator and reasoned why that number could be the common denominator.

The shopkeeper context with the paper bar as a model was a great activity to investigate the common denominator. The paper bar model together with the contextual situation about making a measurement tool was found to be really helpful to set aside the procedural strategies for the students with whom we worked. They were taken to think in a logical manner to find the common denominator through the contextual situation and the help of the paper bar model. For instance, on the one hand, Anas shared his thoughts that he found the common denominator by multiplying the overall corresponding denominators. On the other hand, Rheina came up with a different strategy that was used later as an acceptable explanation by all other students. Rheina’s idea was that she should find a number that could be divided up by all of the given denominators.

I thought I succeeded in guiding the students away from working directly in abstract learning. There was one truly unpredictable question about the common denominator from a student. She wondered about which denominator to choose if several common denominators were possible. This really led to a nice discussion among students.

The paper bar, on the other hand, was a model for the learning of fractions in most of the activities in this research experiment. The paper bar model helped students explain their reasoning. This model was also not too far from the number line one, and therefore useful to explore the idea of relation among fractions (comparing fractions). The paper bar model was also found to be in the students' sense for the moment that they were given a post-test about comparing fractions; most of them used the drawing bar to help them explain their answers and reasoning for comparing fractions. The students always reasoned that the bigger the area of fraction in the bar, the bigger the fraction was. However, two students used the drawing bar as a tool to overcome the comparing fractions problem. They were really convinced to use their drawing as a tool to solve the given problems. In this case, a tool was sort of a ready made model in which students could see the fractions directly. In contrast, their bar drawings did not really resemble fair divisions. Besides, the drawing bar was helpful as well for the students to endeavor finding the equal fractions whenever they found the common denominator in the addition of fractions problem.

In the math congress about addition of fractions, there were two different solutions proposed by two students; the teacher then made a drawing bar facilitating the other students to capture the idea of those two students' explanation. Moreover, in one of the activities, the paper bar model was functioning as an intermediate model approaching the number line model. Clearly, the bar model was not too far different from the number line model. Such a powerful model help the students work within relation among fractions and do some simple operations with fractions.

The paper bar was also used in the paper folding activity, but we could not see any incredible goal achievements of exploring the equivalence of fractions in the research experiment of the paper folding activity. This was due to some missing instructions in the worksheet, and the contextual situation was not strong enough to explore the equivalence of fractions. This may prove that although we used a good model in the learning process, this does not give guarantee that we will see the goal of the achieved learning process. Collaboration between contextual situations that comply with the students' imagination and the related models need be taken into account.

In order to help the students explore equivalence of fractions, therefore, the rubber bands activity, which was related to measurement, was introduced to refine the paper folding activity to achieve the idea of equivalence of fractions. The rubber bands activity that elaborates the cone cap games context of Indonesia Independence Day was familiar to students and the string rubber bands as a model was a good activity to discover equal fractions, both by unit and non unit fractions. The string rubber bands model that can be stretched is an interesting model to explore the equivalence of fractions. It was proved by some students, by interview, that if they had to find the equivalence of fractions, they argued with what they had found in the string rubber bands activity.

The string rubber bands model itself was also not too far different from the number line model. However, this activity did not work well in the research experiment, especially in finding the equivalence of the non unit fractions. This was due to the missing questions about the non unit fractions. Some students had used their logical thinking in exploring the equivalence of fractions within the rubber bands activity by remembering the given contextual situation. However, there were some students who performed different strategies in finding the equivalence of fractions using the string rubber bands that tended to be more procedural. This happened because the teacher directed the students back to the procedural way. The teacher forgot the given contextual situation about the rubber bands activity. Again, we saw the importance of the contextual situation being used to bring students to develop mathematical concepts keeping away from the procedural way.

The next activity to discuss is that of the addition of fractions. In this type of activity, the students applied their previous knowledge about equal fractions and the common denominator. Most of them encountered difficulties in finding the numerator after finding the common denominator. To make it easy, the students drew the bar to find the numerator. Based on the above explanation, the important roles of the contextual situation and the three models related to them in our design research have become clearly elaborated. Also, in our sequential activities in the research experiment, students were encouraged not to apply the procedural ways in which fractions had been introduced to them in grade 3. I observed that they had lost most of their knowledge of fractions from their previous learning. It was found that the students could follow our proposed activities and tend to be more courageous in giving responses and sharing their reasons.

Apart from the above illustration, we analyzed the attained data in the third phase, i.e. the Retrospective Analysis phase. This related to our second research question about the development of the social norms in teaching and learning within the sequence of activities. We found some changes in the students' habit in the classroom. At the beginning, before we came up with our sequence of activities, students barely discussed things among each other, and worked on many mathematical operations by themselves.

They were also afraid of sharing their ideas and reasoning in the discussion since they did not use to do it. In the meantime we did the research experiment; students were getting used to have more discussion with their peers in the class in solving the problems they encountered. In the discussion, they learned to share their ideas and listen to their friends' opinion. The students also learned to reason about their answers to clarify their ideas. In addition, they did not have to stay at their own desk, but rather they could move around in doing the activity. This could motivate them and made the learning process become more fun and avoid the students' boredom in learning.

On the teachers' side, some changes in habit were also found during our research experiments. A teacher explained that before we introduced our Hypothetical Learning Trajectory (HLT II), she felt that she just forced the students to take her strategy as it was to solve problems. She always led the students to at least a semi abstract way of learning rather than learning with concrete things. The teacher used to explain mathematical concepts by writing on the whiteboard and then just had the students copy any writing on the board into their own notebook. Afterwards, they were asked to do some problems from the textbook.

Then, when we interviewed the teacher, she felt that her role in the teaching was only as a mediator and a guide who orchestrated the discussion among students. Within the research experiment, the teacher was used to developing the social mathematical norms in her class by always asking the possible different strategies from other students and having them explore their reasoning.

To cover up the entire set of activities, the three models were used for the students' reasoning and they explained their answers from the advance activities, then commented, criticized and questioned each other in a way that rarely happened in class before. As for the students attempting to reason about their answer, it could be the case that they gained more understanding of the problem. The method of teaching was also changed. The teacher's role was not just to transfer her mathematical knowledge directly to students, but more in guiding and orchestrating students' discussion.

So far, we have suggested both strengths and weaknesses of our design research. From the aforementioned advantages that we could benefit from, as summarized above, our sequence activities are, of course, still open to further constructive revisions for future implementations. Most of the revisions could be on the sides of instruction, worksheets and formulation of questions. Also, the learning lines activity itself needs to be refined. To summarize all, we propose a set of activities described as HLT III on the basis of the analysis from our research findings referring to HLT II for the betterment implementation of learning fraction in classroom setting including some remarks from our research experiment. In the HLT III (see appendices), we can find a different arrangement of the learning lines from HLT II with some revision of each activity. We also provide our remarks and suggestions referring to the result of the research experiment in HLT III.

VI. Discussion

The core of a design research constitutes a cyclic process of designing instructional sequences, testing and revising them in classroom settings, and then analyzing the learning of the class so that the cycle of design, revision, and implementation could begin again [Gravemeijer & Cobb, 2001]. This cyclic design could be started again for a couple of times to improve its implementation in a classroom setting.

In our study, this type of cyclic design has been implemented on fractions. Based on the result of the try out activity in the first phase of the design research, we did some revisions to change the order of the sequence activities and to add some activities here and there. This yielded HLT II that was used as a guide for the research experiment phase. The cycle of the design did not stop at HLT II. There were some revisions of the HLT II based on daily analysis in the research experiment phase. The revisions were continued by considering the analysis of the result of the research experiment based on HLT II. Therefore, a set of activities described as HLT III has been proposed for a better implementation of learning fractions in a classroom setting.

In this design research, there were three contextual situations, i.e. dividing cakes, the cone cap games, and the shopkeeper context. In addition, this study used three models that were related to the given contexts, namely the paper bar, the string rubber bands and the number line model. Accordingly, both of the given contexts and models have raised the first research question as stated in the previous parts: "What roles do contexts, and models that are related to them, play in the teaching and learning of fractions?" With respect to this question, as deeply analyzed in chapter 5, we could come to the conclusion that contextual situations obviously have important roles in the learning process.

At one hand, a context, as the starting point of sequential learning activities modified by the related models, should remain in the students' mind and bring them to engage in the given context. It should not start with formal procedures of Mathematics, but rather stimulate students to understand Mathematics better than that in the procedural or direct abstract ways of learning. If this does still hold, however, it should be done in a way that students should come to understand mathematical aspects beyond the given problems and in a way that they are capable of reasoning to their answers.

On the other hand, the three related models should be collaborated with the given contextual situations that comply with the students' imagination. The first model that was mentioned, the paper bar, is a model for learning fractions in most activities in this research experiment. Also, it is used as a model for thinking and reasoning about relations among fractions (comparing fractions and equivalence of fractions), for finding common denominators as well as for investigating addition of fractions. Moreover, it serves as an intermediate model to approach the number line model. The model is very powerful one to explore the relations among fractions and, further, help students do simple operation with fractions. Another model is the stretchable string rubber bands. This interesting model functions as a generalizing model to explore the equivalence of fractions and is introduced to refine the paper folding activity to achieve the goal of finding the equivalence of fractions.

The second research question was stated as: “How do the social norms in teaching and learning develop within the sequence of activities?”. We expected that the sequence of activities that we proposed in our design experiment, would lead students to have more discussions in solving any problems they encounter. Further, they would also learn to reason on their answers to clarify their ideas. In general, this expectation was confirmed.

In practice, there will, of course, still be some students in a class who find it difficult to reason on their answer. They will feel afraid to make mistakes in answering the given questions and ashamed of committing mistakes to reason. Nevertheless, they could still come up with the final answer of the given questions without explaining their reasoning.

To enhance the students’ self confidence in reasoning and to get them involved a discussion, one possible solution is to let the students work first in a small group. This is done to give students the chance, especially the slow pace ones with less self confidence, to share their ideas in a small group. By doing so, the teacher could also personally approach the students who need further guidance. Then, the learning goes on to a class discussion. In this case, the role of a teacher is to orchestrate a discussion among students and to explore their reasons. A little ’force’ seems to be required for some students to let them get used to reasoning by asking ”Why” and ”How” questions. In additon, a teacher can also provide rewards as a motivation for his/her students who are proactive and capable of sharing their ideas and reasons. This, in turn, can lead to the improvement of students’ habit in learning.

As a reflection of our research findings, there is an important remark to be made about the processes of teaching and learning mathematics. Many teachers still use a traditional way of teaching, trying to transfer their knowledge to the students.

However, the development of Mathematics education still goes on. The reform of learning and teaching mathematics based on RME apparently offers opportunities for students to discuss and construct recognizable contexts. Ultimately, I hope my study will contribute to that development.

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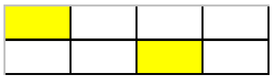
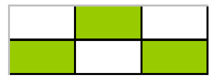
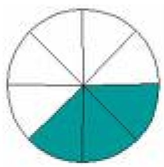
PRE-TEST

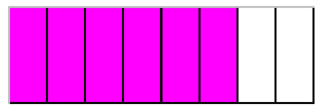
Name :

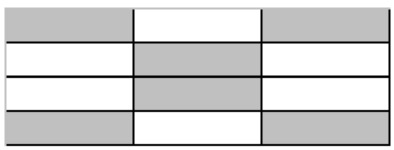
I. Below are given some pictures. Write parts that are drawn with fraction











II. Do you like sausages that were usually eaten with bread?

Sausages were shared to some students in a school. Every student got different parts. Draw the part of sausage that they will get below

Part of sausage	Sausage	How to divide and draw?
Ana gets ¼ sausage		
Reni gets ½ sausage		
Billa gets ¾ sausage		
Amadra gets 4/8 sausage		

III. Filled the fractions in empty train fractions below:

(Note: there is an ordering fraction in the train)

1/5	…..	…..	4/5	…..

POST TEST

Name :

1. Below are given some fraction part of cake. Grouping the equivalence of fraction part of the cake in the oval’s pocket below:

	¾	½	¼	2/3	
3/12	1/3	5/10	6/8	3/9	
	2/4	4/6	4/16	4/12	3/6

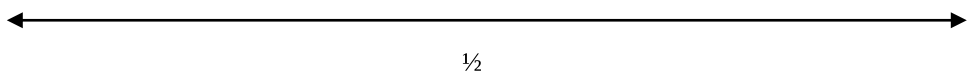
2. Compared the part of cake below (more or less or equivalence). You can explain your answers and reason with words or drawing.

   a. 1/6 cake and 1/8 cake

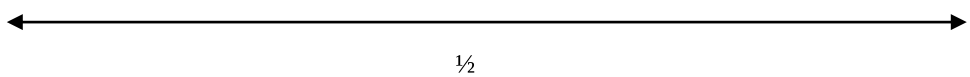
   b. 4/8 cake and ½ cake

c. ¾ cake and 2/3 cake

3. a. Posit ¼ in the number line below



b. Posit 2/3 in the number line below



4. The school's canteen sold two kinds of cakes : chocolate taste and vanilla taste. The price of one whole part of the cake was Rp. 12000,-

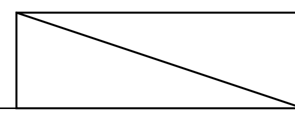
a.Donald Bebek bought 1/6 chocolate's cake and ¾ vanilla's cake. How much part is the total cake that were bought by Donald and How much money that is paid? Explain your answers and your reason with word or drawing.

a. Minnie buys ¾ chocolate's cake and 1/12 vanilla's cake. How much part is the total cake that were bought by Minnie and How much money that is paid? Explain your answers and your reason with word or drawing.

b. Mickey Mouse bought 1/2 chocolate's cake and 1/8 vanilla's cake. How much part is the total cake that is bought by Mickey and How much money that is paid? Explain your answers and your reason with word or drawing.

Write your difficulty in studying fraction before?

The Hypothetical Learning Trajectory III

No	The title of activities and the learning goal	The Activity	Explanation behind the activity	The conjecture of students' strategies and the expectation	Remarks from the research experiment
1	***The Clever paper folding activity which has meaning of fair sharing or measuring and the symbolizing of the paper folding activity*** Learning goal : • Students will explore the equivalence of fractions in the paper folding activity and describe the iteration of unit fractions	First of all, Teacher tells that she is challenged by her father to divide the "Lapis Surabaya" fairly. Students are asked to divide the cake by fold the paper bar below the cake fairly. The paper bar under the cake is shown in front of the class by the teacher. Based on that situation, teacher asks students' help to solve the problem. Below is the picture of Indonesian's cake that is familiar to students.  There will be some problems given to the students. They were asked to divide the cake based on the demand of the teacher's father. Some paper bars represent the cake will be folded by students. 8. Divide into 2 9. Divide into 4 10. Divide into 3 11. Divide into 6	This first contextual activity will guide the students to do the paper folding. The idea is that students fold the pieces or parts of paper bar all have to be in equal size. With this activity, students will be challenged to do "clever folding". Students will understand that dividing into a number means folding the paper into a number. First of all, students are asked to divide the paper bar into two, they will easily to do that. Next they are asked to fold the paper	There are some papers bars given represent cake to be divided fairly. Since the cake is long cake and it could be represented with long paper bar, the fourth grade students probably will use some tools that are familiar for them such as ruler to measure and divide the paper fairly. Let students do what they want first. To face students who use tool to measure and divide the paper, the cake must be made so that it would difficult to be divided use ruler for example. The different strategy will be used by students, they can divide the cake into several shapes such as 	We need to revise the worksheet. The worksheet should not only include a question about the strategy of the students to fold the paper but also add some guiding questions about exploring the equivalence of fractions and also questions related to the iteration of unit fractions to get the non unit fractions. For symbolizing the paper folding activity, let the students to write the symbol nearby

	to get the non unit fractions. • Students will be able to symbolize the paper folding with fraction's symbol, not only with the unit fraction but also with the non unit fraction	12. Divide into 8 13. Divide into 5 14. Divide into 7 Students will work in group of two or three to have discussion between them. After the students do clever paper folding, they symbolize their paper folding with fractions' symbol. The activity of symbolizing the paper folding will help them to understand the iteration of unit fractions. In symbolizing the fractions, teacher guide the students to write the symbol nearby the last edge of the folding. The worksheet is also provided to the students to describe the iteration of the unit fractions to get the non unit fractions and the way how they fold the paper. Interviewing and guiding students could be done in this activity.	bar into four. In folding the paper into 4, they will do it through the previous folding paper strategy (fold the paper into two). They will fold the two part paper into two again, so they will have four parts. The iteration of the unit fractions can be done when they did this strategy. Students will investigate the relation between fractions by this activity for example they could divide a paper bar into 4 parts by first finding "1/2" and then dividing each half into two parts. They will see that 1/2 will be equal to 1/4 +1/4 or 2/4. For symbolizing of the paper folding activity, let students write the	 If we face this condition, we can anticipate with using a cake that usually cut in slices so they will not cut it diagonally like the probably happened above. Other conjecture is that there are students who fold the paper to have fair division. If we found this strategy, we can show it to other students with give advice that it will be handier to do, beside that students can train to have clever paper folding through that strategy. Through taken as shared and use clever paper folding as strategy to fair divide the cake, students do paper folding activity to divide the paper based on the question in students' worksheet. Discussion in group to have clever paper folding is held. Students also will describe their strategy in the student's worksheet.	the last border folded.

<table>
<tr><td></td><td></td><td></td><td>fractions' symbol nearby the last border, it was needed to encounter the misunderstanding in the next activity that is the activity of number line.
The task also includes dividing the paper bar into five and seven. Although our conjecture is that students will find it difficult to divide the paper bar to those numbers, we still put it as a task to let them realize that dividing a paper bar into five or seven is more difficult than to other numbers.</td><td>For the symbolizing of the paper folding activity, probably some student's still use unit fractions instead of non unit fractions. Every part of the paper strip is notated use unit fraction. For example: the paper strip that dividing into 3 , they will notate as below :<table><tr><td>1/3</td><td>1/3</td><td>1/3</td></tr></table>This kind of answers could be used as topics for discussion in the math congress about the iteration of the unit fractions to be the non unit fractions. There are probably also some students who write the non unit fractions in the paper bar.</td><td></td></tr>
<tr><td>2</td><td>Math congress about the clever paper folding and the symbolizing of the paper</td><td>Teacher holds a class discussion in which students share their ideas related to the paper folding activity and the symbolizing of the paper folding activity. Teacher will orchestrate the discussion. Some groups of students were chosen to share their idea and</td><td>The strategy that appeared in the previous activity will be discussed. One topic of the discussion is about the</td><td>I expect that students will share their strategy in folding the paper bar and the symbolizing of the paper folding whether they use the unit fractions or the non unit fractions. They</td><td>Pay more attention to the exploring of the meaning of fractions and focus on the meaning of non unit fractions</td></tr>
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	folding activity Learning goal: students will share their strategies in folding the paper bar, the symbolizing of the paper folding activity, finding the equivalence and iteration of fractions	teacher could ask other students about their friend's strategy. The teacher can bring the students to focus on the importance of dividing the paper bar fairly and the iteration of the unit fractions to get the non unit fractions .	symbolizing of the paper folding that led to the iteration fractions. For students who write the non unit fractions, it will be related to the number line of fractions in the next activity.	will be able to do iteration of the unit fractions to get the non unit fractions.	as the result of iteration of unit fractions (3/4 as three times of ¼). This can be used if students struggle with finding the equivalence of non unit fractions.
3	***Refining the idea of equal fractions using the string rubber bands as model (The rubber bands activity).*** Learning goal: Students will refine their	The New contextual situation derived about game of 17 August to celebrate Indonesian' independent day. The game that will be hold was the cone cap game. Before students play the game, they should help the picture of the game to give overview	The new model was used in the string rubber bands. This model could help students to prepare fair distance of string to put the hanger in it. With the characteristic of the rubber band that could be stretched, it	The student's strategy that probably come up in dividing into some number is that students will make a string of rubber bands, but of course the frustrating thing come up since it will be difficult for them if they have to divide the line into 2 for example since the string rubber bands with two rubbers is not enough to be stretched	Pay more attention to the non unit fractions. We need to add some questions in the worksheet about the equivalence of the non unit fractions

	understanding of the equivalence of fractions through the rubber bands activity.	 For the game, it needs to prepare fair distance of string for place for hang the cone cap. At that moment, there is no rope that can be used but teacher have an idea to use the rubber band since she just see some kids play with string rubber bands. Teacher shows some of the big rubber bands and ask student to make string rubber bands based on how many parts of area that they want to divide. For example: with two rubber bands, they will divide the length of black board in two and put a hanger in the middle of it. Our conjecture was that they can not do it with two rubber bands to have ½ parts. Student will discuss and use 4 or 6 or 8, etc to divide the string. They shall come to the idea that ½=2/4=3/6=4/8 With three rubber bands, they will divide the string in three and put two hangers in every tied rubber band. Students will explore to divide the string also in 4 and 6.	will help student to put the hanger of the cone cap with fair distance. With this new model, students can also solve the problem that are faced by students in the paper folding activity that is dividing a paper into odd number, for example if they want to divide a thing into seven, they can use seven rubber bands and stretch it if the thing was long. With this activity, it helps students refine their understanding of equal fractions. The mathematical discourse that came up through this activity was operations with fraction that can also be done with	that long distance. They shall find that they could use four; six, eight, etc then divide it into two.	

		The teacher also needs to ask the students to show the non unit fractions in the length of something. With the same idea before, they will explore the equivalence fractions of the non unit fractions.	denominator more than 12.		
4	***Math congress of equivalence fractions (string rubber bands activity)*** Learning goal: Students will share their experience and ideas of equivalence of fractions within certain domain (denominator up to 12) and odd denominator	Teacher holds a class discussion. Teacher will orchestrate the discussion. Some group of students will be chosen to share their idea and teacher could ask other students opinion about their friend's strategy so there will be a discussion in class. The activity can be emerged to other unit fractions	Students can share their knowledge related with equivalence of fraction through rubber bands activity.	In sharing knowledge, students show their previous experience when work with rubber bands.	After students explore the equivalence of unit fractions, it needs to ask the equal fractions of non unit fractions, for example by asking the students to show 2/3 of the black board using the string of rubber bands. They will use six rubber bands and tell what the equivalent fractions of 2/3 are.
5	***Explore the relations between fractions in the***	Teacher holds a class discussion about the activity of number line of fractions. Teacher will guide the discussion. Some groups of	Before start with the number line, teacher can hold discussion	The expectation that hopefully appeared from the activity of number line, students will posit	More questions and exploration of the meaning of

	math congress about the activity of number line Learning goal: Students will share their ideas and their experience in symbolizing the paper folding and also discuss the meaning of non unit fractions and the relation between fractions in the number line.	students are chosen to share their idea and teacher can ask to other students about their friend's strategy, so there will be a discussion in the class. There is also a discussion about non unit fraction and its meaning through number line model. The number line that is given is not the drawing of number line on the blackboard but use a rope and some fraction paper that could be hanged on the rope.	about the symbolizing the paper folding that previously done Many kinds of symbolization can be used as reference for students. We gave one of the example of the symbolization below:. | 1/3 | 2/3 | 3/3 | If there are students who used that kind of symbolization, teacher can discusses the meaning of fractions in it and in the number line model, for example: 2/3 is the name of just the second part or that was the name of the first two parts together? The explanation can be drawn in the number line. Students will put the fraction in the	the fraction in the number line. With the number line, students will show the equivalence of fractions and comparing fractions. Students can reason that the bigger fraction, the more to the right was its position. For the equivalence of fractions, the fraction cards that are put in the same position are equal fractions.	fractions and the relation between fractions (comparing fractions) are needed in the number line.

			number line and explain the meaning of it. Through this activity, students do not only understand the unit fraction, they are also trained with non unit fraction. The different strategies from students will appear in the class discussion. Teacher will guide students to get the idea of equivalence of fractions from the paper folding activity.		
6.	***Expand to the idea of a common denominator through the shopkeeper context*** Learning goal:	The next contextual problem is appeared. The teacher continues her story about shopkeeper who sells long cake. The shop provides the long cake and sells it in small parts. Teacher tells story that the shop was very crowded everyday. The shop sells 1/3, 1/2, 2/3, ¼, ¾ and 1/6 cake. In the previous activity, students already made some paper folding of the part of the cake	This contextual situation will guide students to the idea of the common denominator. Students do not only work with unit fractions, they also work with non unit fractions. Part of	By discussion, students face some common denominator from many fractions' combination. For example, if they have paper bar for ¼, they don't need paper bar for ½ (the idea is that 4 was the common denominator of fraction with denominator 2	

	Students will share their ideas and their experiences in the problem of helping the shopkeeper, related to the idea of the common denominator	above and the teacher show it to the shopkeeper. But, the shopkeeper asks whether he really needs all the bars as the tool to help the shopkeeper cut the cake? The teacher can also ask whether it was possible to just make one bar as helper cutting tool. Let students discuss in a group and the paper folding can be provided if it is needed by students. Beside that, worksheet is provided to record the thinking process of students.	the cake that is given is with denominator 2,3,4,6.	and 4. If they have paper bar with fraction with denominator 6, they don't need paper bar for ½ and 1/3 (the idea was 6 is the common denominator of fraction with denominator 2 and 3). By the last questions, we will guide students to common denominator 12 (for fraction with denominator 2,3,4,6)	
7	***Math Congress about the common denominator activity (shopkeeper context)*** Learning goal: The Students share their knowledge related to the idea of common denominator	Teacher holds a class discussion. Teacher will orchestrate the discussion. Some group of students will be chosen to share their idea about the previous activity and teacher could ask other students opinion about their friend's strategy so there will be a discussion in class. The teacher will provide some fractions on the white board and ask the students the common denominator of those fractions.	Students could share their knowledge related with the idea of common denominator	Class discussion	We need to add a discussion about the chosen denominator (The least common denominator) if we have some common denominators.

8	***Develop the context of cake shop to explore the addition of fractions*** Learning goal: Students will be able to solve the addition of fractions in a certain domain (denominator up to 12)	This activity is still related to the shopkeeper context. The cake shop sells two types of cake (chocolate and vanilla). The price of both cakes is same Rp. 12.000,-. There are so many customer who want to buy two parts of cake, for example: 1/3 chocolate and ½ vanilla 2/3 chocolate and ¼ vanilla 1/6 chocolate and ½ vanilla ¾ chocolate and 1/12 vanilla To make the shopkeeper work as fast as possible, students are asked to look for the total part of cake and the total price that should be paid. Let students discussed this problem in group. The worksheet is also provided to record their strategy in helping the shopkeeper.	This activity will explore the addition of fractions.	There are some possibilities, students directly count the price of every part of the cake, for example by calculating every part of the cake by the price and then add it. The first level of students: students will posit the symbolized bar which was explored before. By posit it and see the total part from 12 parts, they could know the total part that will be sold.	
9	***Math congress about the activity of the addition of fractions (cake shop context)*** Learning goal: Students can share their understanding and their strategies to find the result of the	Teacher held a class discussion. Teacher will orchestrate the discussion. Some group of students will be chosen to share their idea and teacher could ask other students opinion about their friend's strategy so there will be a discussion in class. Teacher writes some addition of fractions' questions on the white board and asks some students to share their idea.	Class discussion	There was a discussion about the chosen of the common denominator by the students. Probably there are still some students who struggle with finding the equivalence fractions of the non unit fractions. To face the struggle, teacher can use the bar as the model to help them solve the problem or remind them to the previous activity related to the	Remind the teacher not to write the addition of fractions problem directly. Don't forget about the price that also should be added.

	addition of fractions.			equivalence of fractions.	