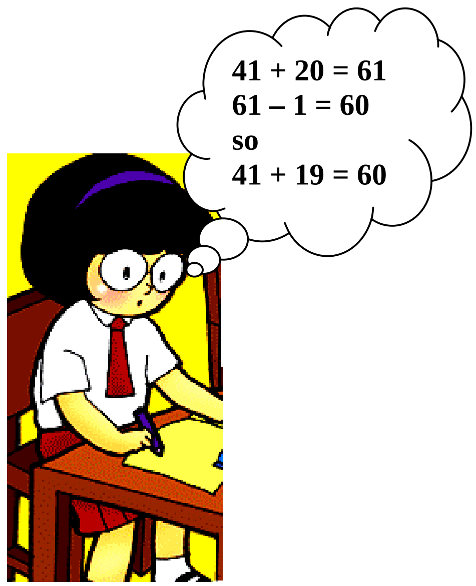
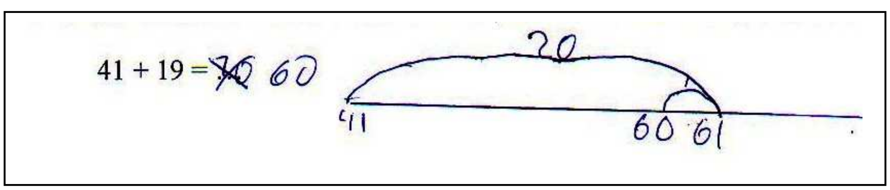
Design Research

on Addition Up to One Hundred using Mental Arithmetic Strategies on an Empty Number Line





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Abstract

Aiming at the development of theory and the improvement of practice about both the process of mathematical learning and the means designed to support that learning, the present design research attempts to answer the question, how the development of children's learning process in constructing mental arithmetic strategies on an empty number line to solve addition problems up to one hundred. The question was answered by taking into account a design research methodology with two underpinning theories, i.e. (1) the Realistic Mathematics Education theory as a guide to develop a local instruction theory in a specific mathematical domain; and (2) the socio-constructivist analysis of instruction in which children's learning processes were viewed from both the individual and the social perspective. The surprising result shows that some children involved in the teaching experiment interpreted an empty number line as a newly taught calculation procedure, so that the empty number line becomes meaningless for the development of children's mental arithmetic strategies. However, it is also shown that children develop flexibility in solving addition problems up to one hundred using mental arithmetic strategies on an empty number line. In addition to this, the empty number line is proved to have a level raising quality as a model for reasoning that could enhance more sophisticated strategies.

Key Words: design research, addition up to one hundred, children's mental arithmetic strategies,
an empty number line, mathematics teaching and learning in elementary school.,
realistic mathematics education

1. Introduction

In recent years, researchers in mathematics education have become increasingly interested in mental arithmetic as a new breakthrough that must precede algorithm in doing calculation for elementary school children (Treffers, 1991; Beishuizen, 1993; Reys, 1995; Klein, 1998). Mental calculation is required while dealing with daily life situation: for instance, it is meaningless to perform a pen and paper algorithm to calculate the amount of money needed to pay for two kilos of sugar and a box of chocolate in the market. Treffers (1991) argued that algorithm is one of the causes of innumeracy[1] in primary schools when it is taught prematurely without context problems, while mental arithmetic strategies[2] and estimation in a realistic approach are suggested as an alternative. Moreover, Hope, *et al* (1988) described these benefits of doing mental calculation:

1. Calculating in your head is a practical life skill
2. Skill at mental math can make written computation easier or quicker
3. Proficiency in mental math contributes to increased skill in estimation
4. Mental calculation can lead to a better understanding of place value, mathematical operations, and basic number properties.

However, mental arithmetic strategies must be introduced thoughtfully with rich contextual situations, in which children have freedom to develop their understanding under the guidance of the teacher. In addition to this, Gravemeijer (1994a) pointed out that an empty number line is found to be a powerful model to do mental arithmetic strategies flexibly and to foster the development of more sophisticated strategies, but which could represent children's informal strategies at the same time. Moreover, Klein (1998) came to the conclusion that providing children with a powerful model like the empty number line, establishing an open classroom culture in which children's solutions are taken seriously, and making teachers aware of both cognitive

[1] Inability to handle numbers and numerical data properly and to evaluate statements regarding sums and situations which invite mental processing and estimating is defined as Innumeracy (Treffers, 1991).

[2] Mental arithmetic is characterized by working with number values instead of number digits, in which a framework of number relations and number sense play important parts in doing mental arithmetic. (Buys, 2001)

and motivational aspects of learning, will help every student become a flexible problem solver. Therefore, contextual situations, the use of model, the proactive role of the teacher and the classroom culture play a crucial role in the development of students' learning in a classroom community.

Although considerable research has been done in many countries, problems still remains in Indonesia, where the mathematics elementary school curriculum's objectives focus on the use of algorithm to solve double-digit addition and subtraction problems since the first grade. Furthermore, Indonesian classroom culture tends to perceive a classroom community with teachers as the central point and children are expected to be obedient in every manner. This will lead to the limitation of children's freedom in doing mathematics in their own way.

On the other hand, a progressive innovation program, i.e. PMRI (Indonesian Realistic Mathematics Education), that has been running for more than six years, has a primary aim to reform mathematics education in Indonesia. This innovation program is adapted from RME (Realistic Mathematics Education) in the Netherlands that views *mathematics as a human activity* (Freudenthal, 1991) in which students build their own understanding in doing mathematics under the guidance of the teacher. In contrast to traditional mathematics education that used a ready-made mathematics as a starting point for instruction, RME emphasizes mathematics education as a process of doing mathematics in reality that leads to a result, mathematics as a product. Sembiring, *et al* (2008) summarized from all RME studies in Indonesia that the RME approach could be utilized in Indonesia and stimulate reform in mathematics education. Therefore, the PMRI-classroom, in which this design research is conducted, has started to build an open classroom culture where children learn to come up with different strategies in solving problems and teachers give more freedom and less instruction in their teaching.

Aiming at the development of theory and improvement of practice about both the process of learning and means designed to support that learning, the present research attempts to answer these questions:

1. How do Indonesian children who are used to perform an algorithm strategy develop their mental arithmetic strategies on an empty number line to solve addition problems up to 100?
2. How does the emergence of an empty number line through a measurement context support the development of the children’s thinking in constructing number relations to construe mental arithmetic strategies?
3. How do the role of the teacher and the classroom culture support the development of the children’s thinking in constructing mental arithmetic strategies?

These questions will be answered by taking into account a design research methodology with two underpinning theories, i.e. (1) the Realistic Mathematics Education theory (Freudenthal, 1991; Treffers, 1987; Gravemeijer, 1994b) as a guide to develop a local instruction theory in a specific mathematical domain; and (2) the socio-constructivist analysis of instruction (Cobb & Yackel, 1996; Cobb *et al*, 2001) where children’s learning processes are viewed from both the individual perspective and the social perspective.

2. Theoretical Framework

In this chapter, we start with the domain-specific instruction theory of RME (Realistic Mathematics Education) followed by the current situation in mathematics teaching and learning in Indonesian elementary schools. We continue by elucidating addition for numbers up to 100 and mental arithmetic strategies. We move on to the three design heuristics of RME in relation to addition up to 100 using mental arithmetic strategies on an empty number line. Finally, we expound more on specific research questions in this design research.

2.1. Realistic Mathematics Education (RME)

Realistic Mathematics Education (RME) is a theory of reformed mathematics education in reaction to the behaviorism[1] perspective and the 'new math'[2] movement during the second half of the 20th century. RME was first developed in the 1970s in the Netherlands by Hans Freudenthal, who viewed mathematics as a *human activity* (Gravemeijer & Terwel, 2000). In contrast to traditional mathematics education that used a ready-made mathematics as a starting point for instruction, RME emphasizes mathematics education as a process of doing mathematics in reality that leads to a result, mathematics as a product.

> [Mathematics as a human activity] is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be a *matter from reality* which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a *mathematical matter*, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach (Freudenthal 1971: 413-414 in Gravemeijer & Terwel 2000)

This clarifies that 'realistic' term in RME doesn't merely means 'a matter from reality' that is always encountered in a daily life situation, but also 'a

[1] Behaviorism theory only measured observable behaviors produced by a learner's response to a stimulus. The behaviorist is not concerned with how or why knowledge is obtained.

[2] The new math emphasized mathematics as an abstract deductive system which was taught in American schools during the 1960s. This was complained about by parents and teachers because the new mathematics curriculum was too far outside students' ordinary experience.

mathematical matter' of an abstract mathematical problem which is meaningful and experientially real for students. However, mathematics education for young children should begin with the organizing and structuring activity–the so called '*mathematizing*' of everyday reality (meaningfully contextual situation) because children haven't encountered a mathematical matter yet that is experientially real to them.

Treffers (1987) distinguished *horizontal* and *vertical* mathematizing that characterizes RME theory. Horizontal mathematizing is transforming a problem field into a mathematical system. It leads from the world of life (a matter from reality or a mathematical matter) to the world of symbols. On the other hand, vertical mathematizing is processing within the mathematical system in which symbols are shaped, reshaped, and manipulated. Progressive mathematizing, both horizontally and vertically, is inspired by five educational tenets as follows.

Five Tenets of RME

Treffers (1987) has defined five tenets for Realistic Mathematics Education:

1. *Phenomenological exploration.*

   Freudenthal's view of mathematics as a human activity requires an extensive phenomenological exploration that is aimedat acquiring a rich collection of contextual situation in which mathematical activities take place. The contextual situation is not merely a word problem, but it has to deal with a reality that makes sense for children with different levels. It should provide an ample space for children to build their own understanding and serve as a basis for model development.

2. *Using models and symbols for progressive mathematization.*

   The development from intuitive, informal, context bound notions towards more formal mathematical concepts is a gradual process of progressive mathematization. A variety of models, schemes, diagrams, and symbols encourages children in this gradual process.

3. *Using students' own construction and productions*

   Students' own constructions deal with their *actions,* while their own productions deal with their *reflection.* This is, in fact, one of the most characteristic features of progressive mathematization in RME instruction theory. Students have freedom to construct their own path in learning under the teacher's guidance. Furthermore, students' own productions as the result of instruction functions as the mirror image of the teacher's didactical activity. In this case, the teacher should help students to build on their understanding from what students know.

4. *Interactivity.*

   The learning process of students is not merely an individual process, but it is also a social activity where students build on their understanding through discussions, collective work reviews, evaluation of various constructions on various levels (comparing strategies), and explanation by the teacher. The interactivity means that students are also confronted with the constructions and productions of their fellows, which can stimulate them to shorten their learning path and to become aware of, in this case, more sophisticated strategies in solving addition problems up to 100.

5. *Intertwinement.*

   The mathematical domain in which students are engaged in should be considered related to other domains, the intertwining of learning strands. In this design research the 'calculation for numbers up to 100' through the measuring activity is intertwined with 'measuring conceptions'. Students develop their understanding of measuring conceptions as well as their counting strategies in the measuring context.

In addition to these, the three design heuristics of RME, i.e. *guided reinvention*, *didactical phenomenology*, and *emergent modeling* will be elaborated in section 2.5, where the relation to addition up to 100 using mental arithmetic strategies on an empty number line is made explicit.

2.2. Mathematics Teaching and Learning in Indonesian Elementary School

The present curriculum in Indonesia – curriculum 2006 (KTSP) – states that mathematics should be given to all students from elementary schools (sekolah dasar) to provide students with the ability to think logically, analytically, systematically, critically and creatively, and with the ability to cooperate. Moreover, the curriculum mentions that mathematics learning should be started from a contextual situation which leads to a mathematical concept under progressive guidance.

In spite of this, in daily practices teachers are still dominated by a traditional 'chalk and talk' lesson structure without a contextual situation. The traditional approach put teachers in a central role where students become passive learners without a great deal of thought in doing mathematics. The teaching and learning of mathematics in Indonesian schools became mechanistic, with teachers tending to dictate formulas and procedures to their students (Armanto, 2002; Fauzan, 2002; Hadi, 2002). As a result, students often have difficulties to comprehend mathematical concepts and to construct and solve mathematical representation from a contextual problem, and the teaching style makes mathematics more difficult to learn and to understand (Sembiring *et al,* 2008).

At this time, an innovation program to improve mathematics teaching and learning in Indonesian schools – PMRI, an Indonesian adaptation of RME – has been running for several years. The main objective of IP-PMRI (Institut Pengembangan Pendidikan Matematika Realistik Indonesia)[3] is to improve the quality of mathematics education in Indonesia through a reform of school mathematics learning theory using realistic mathematics education (RME). RME principles in which students are guided to reinvent mathematics in their way starting from a contextual situation is in line with the current thinking about mathematical learning in Indonesia that emphasizes student-active learning, problem-solving and the application of mathematics in daily life.

Developing student materials for elementary schools, training for teachers to implement PMRI in their classroom, developing contexts and activities to enrich student activities in classrooms, and facilitating RME education for mathematics educators in associated institutes are part of the PMRI programs. These programs have brought many benefits, not only for students, but also for teachers who are involved. Students become more enthusiastic in doing mathematics, more flexible in solving problems and better able to express their thinking. Teachers have changed their mathematics teaching approaches as a result of their involvement with new materials, textbooks, investigation, experiments, in-service education and in-class training (Sembiring *et al,* 2008).

[3] (http://pmri.or.id/)

2.3. Addition for Numbers Up to One Hundred

This section provides descriptions of children's conceptualizations for multi-digit addition, and children's two-digit addition methods in accordance with the level of addition operations for numbers up to one hundred.

Fuson *et al* (1997) distinguished three different multi-digit addition and subtraction conceptualizations for children:

1. Children may add or subtract multi-digit numbers within the counting word sequence (i.e., use *sequence methods* such as counting on by tens and ones).
2. Children may add or subtract the multi-digit numbers directly (i.e. use *collected multiunit methods* such as count or add the hundreds, then count or add the tens, then count or add the ones).
3. Children may also conceptualize such problems as involving *concatenated single digits* and operate as if they were adding and subtracting separate columns of single digits. This conception is used especially with vertical numeral problems which lead to many different errors.

Based on our observations and Indonesian curriculum objectives on operation of numbers, it appears that mainly Indonesian children are taught the *concatenated single digit* conception that leads to algorithm (vertical column) strategy to solve arithmetical problems. However, we could still minimize the error prone caused by this conception of multi-digit addition by investigating and developing various methods that might come out from children informally.

The next section summarized Fuson *et al* (1997) and Treffers & Buys (2001) on different methods of two-digit addition in accordance with the level of addition operations for numbers up to one hundred:

1. *Unitary methods* (the level of calculation by counting)

   Children can make objects for each number and count all the objects. At the next level, children can count / add on by ones from the first number or the bigger number. However, the difficulty is that children have to keep track accurately of how many they have just added while counting. For example: 48 + 29 calculated as: (48), 49, 50, …,72, 73, 74, …, 77

2. *Methods using tens* (the level of calculation by structuring)

   In calculation by structuring, children often make use of the two basic skills in varied ways: the jump-of-ten and the jump-via-ten. One can either start by jumping to the nearest multiple of ten or start by making a jump of ten.

   - *Begin-with-one-number methods*

     This method requires not only counting by tens (30, 40, 50, etc) but also counting on from an arbitrary number (e.g. 38, 48, 58, etc). It is argued that this method is easy to do when objects or drawings or numbers are used to keep track of the counting or adding. For example:

     48 + 29 = (48+20)+9= 68+9 = 77 (jump-of-ten)

     48 + 29 = (48+2)+20+7 = 50+20+7 = 70+7 = 77 (jump-via-ten)

   - *Mixed methods*

     This method begins with separating both numbers, adding the tens, and then moving into the sequence by adding the original ones, and then adding the other ones.

     For example: 48 + 29 = (40 + 20)+8+9=60+8+9=68+9=77

   - *Change-both-numbers methods*

     This method can be easily confused if children do not understand what must remain the same in each method. In addition, the total must stay the same; this method can be thought of as just moving some entities from one number to the other to make one number easy to add.

     For example: 48 + 29 = (50 + 30) – 2 – 1 = 80 – 3 = 77

   - *Decompose-tens-and-ones methods*

     The methods in which the tens and ones are decomposed and then operated on separately must deal explicitly with regrouping: in adding, a unit of ten must be made from ten ones, and in subtracting, a unit of ten must be opened to make ten ones.

     For example: 48 + 29 = (40+20) + (8+9) = 60 + 17 = 77

3. The level of flexible, formal calculation

   At this level, children no longer need to use this or any other kind of visual counting aid as they can carry out calculations entirely in their heads, noting down intermediate steps where necessary.

2.4. Mental Arithmetic Strategies

According to the Merriam-Webster dictionary[4], *mental* can be defined as occurring or experienced in the mind; *arithmetic* is a branch of mathematics that deals usually with nonnegative real numbers and the application of the operations of addition, subtraction, multiplication, and division to them; and a *strategy* can be defined as a careful plan or method. Therefore, mental arithmetic strategies can be defined literally as methods that occur in the mind to solve addition, subtraction, multiplication or division problems with nonnegative real numbers. However, there is a significant difference between what is known as mental arithmetic strategies in Indonesia and what is defined as mental arithmetic strategies here.

Mental arithmetic for doing calculation mentally is currently a new trend in Indonesia. There are two different types of this current trend for mental arithmetic in Indonesia. The first type refers to the '*abacus mental arithmetic*'[5] which was first developed in China. The abacus – a wooden frame with beads sliding on wires – is a calculating tool for performing arithmetic by sliding beads up and down along wires. Children use a concrete abacus as a calculating tool only in the beginning. At the next level, children move their fingers very fast as if they move beads on an abacus while doing the mental calculation in their head. As the primary aim of this method is to solve bare arithmetic problems as fast as possible, it often neglects contextual problems.

The second type is the so-called '*kumon*'[6] in which children learn mathematics starting from their own pre-knowledge. In developing mental arithmetic strategies for children, this method develops number relations by practicing based addition and subtraction problems for numbers up to 20. It also applies number structures to develop number relations and to practice solving addition problems mentally. This method comes to an end of solving addition problems in a mentally algorithmic procedure that treats numbers as *concatenated single-digit conception.*

[4] http://www.merriam-webster.com/
[5] http://www.aritmatikaindonesia.com/index.html
[6] http://www.kumon.co.id/

The mental arithmetic strategies defined in the present research aim at understanding number relations to perform mental calculation, not only bare number problems, but also contextual problem situations, and draws on Buys (2001) definition:

> Mental arithmetic is a way of approaching numbers and numerical information in which numbers are dealt with in a handy and flexible way, and characterized by:
> - working with number values instead and not with digits; in mental arithmetic the numbers keep their value
> - using elementary calculation properties and number relationships such as
>   exchange property (16 + 47 = 47 + 16)
>   splitting property (28 + 43 = (20 + 40) + (8+3))
>   inverse-relationship (62 – 59 = 3, because 59 + 3 = 62)
>   and their combinations
> - being supported by a well-developed feeling for numbers and a sound knowledge of elementary number facts in the area up to twenty and up to one hundred
> - possibly using suitable intermediate notes according to the situation, but mainly by calculating mentally (Buys, 2001: 122).

Buys (2001) argued that mental arithmetic is not only important in our everyday life, but also important as the basis for arithmetic activities in different subject areas. Furthermore, Buys (2001) described three elementary forms of mental arithmetic in general:

1. Mental arithmetic by a *stringing* strategy

   The numbers are primarily seen as objects in the counting row and for which the operations are movements along the counting row: further (+) or back (-), repeatedly further (x), or repeatedly back (:). In this strategy children might conceptualize an addition problem as *sequence methods* in which they could employ a *unitary method* to count on from the first number, or *a method using tens*, for example ‘jump-of-ten’ or ‘jump-via-ten’ strategy.

2. Mental arithmetic by a *splitting* strategy

   The numbers are primarily seen as objects with a decimal structure and in which operations are performed by splitting and processing the numbers based on this structure. In this strategy children conceptualize an addition problem as *collected multiunit methods.* Children have to decompose the ‘tens’ and ‘ones’ and operate them separately. Based on previous research, children normally apply this strategy more than other strategies.

3. Mental arithmetic by a *varying* strategy
   This strategy is based on arithmetic properties in which the numbers are seen as objects that can be structured in all sorts of ways and in which operations take place by choosing a suitable structure and using the appropriate arithmetic properties.

In choosing a model to support mental arithmetic strategies, it is important to find one which is widely applicable – one which can connect with the various manifestations of the operations. Numbers can be structured as a line model that offers more potential than a group model. A group model has a less obvious relationship to a linear situation such as page numbers or people's ages or with one-by-one counting. Moreover, grouping doesn't fit in so well with the informal strategies that children naturally use in these types of situations. It is the proper combination of line and group models that is most productive.

2.5. Three Design Heuristics of RME in relation to Addition up to 100 using Mental Arithmetic Strategies on an Empty Number Line

The developers of RME take the view that RME was not contrived as a formal, finished, instructional theory. Instead, RME is a theory in progress that is always reconstructed as a generalization over numerous local instruction theories. (Gravemeijer *et al*, 2000). RME emerges in a cyclically cumulative process of a series of design research projects (Gravemeijer, 1994, 2004). Therefore, the theory development takes place at various levels: (Gravemeijer, 2004; Gravemeijer & Cobb, 2006)

- the instructional activities (micro theories) level
- the instructional sequence (local instruction theories) level
- the domain-specific instruction theory level (RME theory)

The general tenets that stem from generalizations of local theories can be viewed as heuristics for instructional design (Gravemeijer *et al,* 2000), which are: *didactical phenomenology, guided reinvention,* and *emergent modeling.*

Freudenthal (1983) distinguished thought objects (*nooumena*) and phenomena (*phainomena*). *Nooumea* are mathematical objects of which a part can be experienced as a *phainomenon*. By means of phenomenology of mathematical concepts, structures, and ideas, the relation between *nooumena* and *phainomena* is organized. Since mathematical concepts, structures, and ideas serve to organize phenomena, their phenomenology indicates which phenomena can be created and extended to organize. **Didactical phenomenology** of a nooumenon is the didactical element on how the relation between the nooumenon and phainomea is acquired in a learning-teaching process.

Didactical phenomenology can serve as a heuristic for designing children's activities that encourage children to develop their mental strategies. As a starting point, experientially real contexts that allow a wide variety of solution procedures should be provided for progressive mathematization. In this respect, this design research builds and extends on the contextual situation of the previous research, which has dual goals, i.e. students' development of measuring conceptions and arithmetical reasoning with numbers less than 100 (Gravemeijer, Bowers, Stephan, 2003). The "measuring situation" is chosen as the contextual situation in this design research because it allows students to start their learning path from different levels and serves as a basis for the emergence of an empty number line as a model for reasoning in calculating with numbers up to 100.

The second design heuristics is **Guided Reinvention** in which mathematics is seen as a human activity and children learn mathematics in their own learning route. A designed learning path has to provide spaces for children to develop their own thinking with the guidance of the teacher. The role of the teacher plays an important role to enhance children's thinking through progressive mathematization.

Learning is not only seen in an individual perspective, but also in a social perspective where the individual participates in and contributes to the development of mathematical practices and the learning community supports the development of children's thinking. By comparing strategies, whole-class discussion of the more efficient solution methods and by the guidance of the teacher, for example to create an open classroom culture, pose questions, and record children's mental strategies, using an empty number line could enhance the development of flexible mental strategies in children's thinking.

Emergent modeling encourages children in the process of gradual growth from experientially real contexts to formal mathematics (progressive mathematization). An empty number line as a model for arithmetical reasoning with its level raising qualities is proposed as a didactical tool for addition and subtraction up to 100 (Treffers, 1991). The empty number line emerges from the model *of* the situation to the model *for* mathematical reasoning in solving addition and subtraction problems using flexible mental strategies. It can be used to describe a solution of such problems by marking the numbers involved and drawing the 'jumps' that correspond with the partial calculation. The advantage of an empty number line is its potential to foster the development of more sophisticated strategies.

2.6. Research Questions

1. How do Indonesian children who are used to perform an algorithm strategy develop their mental arithmetic strategies on an empty number line to solve addition problems up to 100?
   1.1. How do children develop a framework of number relations to construe flexible mental arithmetic strategies?
   1.2. How flexibly do children use mental arithmetic strategies on an empty number line to solve addition problems up to 100?
   1.3. Do children interpret an empty number line as a newly taught calculation procedure?
2. How does the emergence of an empty number line through a measurement context support the development of children's thinking in constructing number relations to support mental arithmetic strategies?
   2.1. How does the measurement context with a string of beads support the development of children's counting strategies and their number relations?
   2.2. How does a pattern of a string of beads support the development of children's counting strategies and number relations in their thinking?
   2.3. How does an empty number line support the development of number relations in children's thinking?
   2.4. How does an empty number line support the development of mental arithmetic to solve addition problems up to 100?
3. How do the role of the teacher and the classroom culture support the development of children's thinking in constructing mental arithmetic strategies?

3. Design Research Methodology

Our methodology falls under the general heading of “design research” which was first proposed as “developmental research” by Freudenthal in the Netherlands to develop the so-called domain-specific instruction theory of RME (Gravemeijer & Cobb, 2006; Freudenthal, 1991).

> Developmental research means:
> Experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and this experience can be transmitted to others to become like their own experience. (Freudenthal, 1991: 161)

The purpose of this design research is to develop a class of theories about both the process of learning and the means designed to support that learning, be it the learning of individual students, of a classroom community, of a professional teaching community, or of a school or school district viewed as an organization (Cobb *et al*, 2003). Edelson (2002) proposed three reasons for engaging in a design research as a form of educational research:

1. Design research provides a productive perspective for theory development because the goal directed nature of design provides a natural focus for theory development.
2. Results of design research which encompass design of activities, materials, and systems are the most useful results regarding the improvement of the education system as the ultimate goal of educational research.
3. Design research directly involves researchers in the improvement of education because researchers have the freedom to explore innovative design. This is in line with the educational system that calls for true innovation.

Basically, a design research has three essential phases, which are the design and preparation phase (thought experiment), the teaching experiment phase (instruction experiment), and the retrospective analysis phase (Gravemeijer & Cobb, 2006; Cobb *et al.*, 2003). Each of these forms a cyclic process both on its own and in a whole design research. Therefore the design experiment consists of cyclic processes of thought experiments and instruction experiments (Freudenthal, 1991).

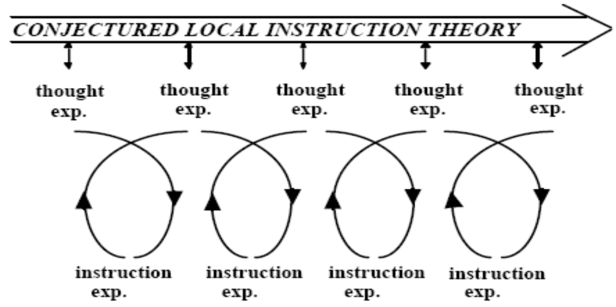


Figure 3.1. Reflexive relation between theory and experiments (Gravemeijer & Cobb, 2006)

In the first phase of this design research, a conjectured local instruction theory is developed under the guidance of the domain-specific instruction theory RME, and then put to the test in the teaching experiment phase, and finally the conjectures are either proved or disproved in the analysis phase to reconstruct the local instruction theory. In this respect, the conjectured local instruction theory guides the cyclically teaching experiment phase while the experiment contributes to the development of the local instruction theory.

3.1. Phase 1: Preparation and Design

The objective of the preliminary phase from a design perspective is to formulate a conjectured local instruction theory that can be elaborated and refined while conducting the experiment, while a crucial issue to highlight from a research perspective is that of clarifying the study's theoretical intent (Gravemeijer & Cobb, 2006). Therefore, a conjectured local instruction theory in the mathematical domain of addition up to 100 using mental arithmetic strategies on an empty number line was designed by first expounding the theoretical framework, then elucidating mathematical learning goals as well as anticipatory thought experiments in which sequences of learning activities and means are designed to support the development of students' thinking. In addition to this, students' mental activities and their levels of thinking in engaging the activities were envisioned.

In order to be able to develop a conjectured local instruction theory, the instructional starting points have to be taken into consideration. Some aspects of these starting points are studying existing research literature and using the results of earlier instructions to envision instructional activities and conjecture children's learning processes. Carrying out pre-assessment before the teaching experiments, such as interviews with the teacher and children, and whole class performance assessments are useful in documenting instructional starting points.

3.2. Phase 2: Teaching Experiment

The second phase is actually conducting the design experiment itself with an aim to improve the conjectured local instruction theory that was developed in the first phase, by testing and revising conjectures as informed by ongoing analysis of both students' reasoning and the learning environment (Gravemeijer & Cobb, 2006; Cobb *et al.*, 2003). The teaching experiment was conducted in grade 2, SDN Percontohan Kompleks IKIP, Jakarta, Indonesia with 37 children in class 2A and Ratna Warsini as the teacher. During the teaching experiment, the researcher also acted as an observer together with the teacher. We used activities and types of instruction that seemed most appropriate at that moment according to the Hypothetical Learning Trajectory (chapter 4). A set of teacher manuals is used as a guide for the teacher in conducting the lessons and also in observing the development of classroom practices (see appendix 1 on the examples of the teacher manual). The data (see appendix 2 on Data Generation) such as video recordings, students' work, and field notes were collected in every lesson, whereas the students' assessments were held before and at the end of the experiment (see appendix 3 on Assessments). The role of the teacher and the classroom culture are also important aspects in conducting the teaching experiment (see appendix 4).

3.3. Phase 3: Retrospective Analysis

A primary aim when conducting a retrospective analysis is to place the design experiment in a broader theoretical context, thereby framing it as a paradigm case of the more encompassing phenomena specified at the beginning (Cobb *et al.*, 2003). The retrospective analysis deals with a set of data collected during the teaching experiment where the HLT was compared with students' actual learning. The HLT functions as guidelines determining what the researcher should focus on in the analysis. The results of the retrospective analysis will form the basis for adjusting the HLT and for answering the research questions.

3.4. Reliability and Validity

A design research can be characterized as a learning process of a research team. The learning process has to justify the results (the HLT including its instructional activities) of the design research. In relation to this, we refer to the methodological norm of reliability (virtual replicability) and validity.

Virtual replicability refers to reliability or reproducibility for a qualitative research in which the research is reported in such a manner that it can be retraced, or virtually replicated by other researchers (Gravemeijer & Cobb, 2006). A criterion for virtual replicability is 'trackability' (Gravemeijer & Cobb, 2001). This means that the reader must be able to track the learning process of the researchers and to reconstruct their study: failures and successes, procedures followed, the conceptual framework used, and the reasons for making certain choices must all be reported. Furthermore, *internal reliability* can be interpreted as inter-subjective agreement among the researchers on the project.

Meanwhile, the design research should provide a basis for adaptation to other situations by offering a *thick* description (results) of what happened in the classroom (Gravemeijer & Cobb, 2006). That is what we mean by the *ecological validity* of this design research. By describing details of the teaching learning process, the role of the teacher, the children's participations and diverse ways of reasoning, the role of model (in this case the empty number line), and the analysis of how these elements may have influenced the whole process, other teachers and researchers will have a basis for deliberating adjustments to other situations. That is to say, a local instruction theory functions as a frame of reference for teachers who want to adapt the corresponding instructional sequence to their own classrooms, and their personal objectives (Gravemeijer & Cobb, 2006). Moreover, *internal validity* concerns the correctness of the findings within the actual research situation. Researchers can improve the quality of their justifications and interpretations by seriously searching for counterexamples or alternative explanations or asking fellow researchers to play the role of 'devil's advocate'.

4. The Hypothetical Learning Trajectory

The aim of the present design research is to develop a local instruction theory on addition up to 100 under the guidance of the domain-specific instruction theory (RME theory), that serves as an empirically grounded theory on how a set of instructional activities can work, not merely to provide teachers with instructional activities that work in the classroom.

> A local instruction theory consists of conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process. The means of support encompass potentially productive instructional activities and (computer tools) as well as an envisioned classroom culture and the proactive role of the teacher. (Gravemeijer & Cobb, 2006: 21)

The conjectured local instruction theory on addition up to 100 which has been developed in the first phase of this design research was put to the test in the teaching experiment phase. The instrument in the present research that can bridge the gap between the instruction theory and teaching experiment is what Simon (1995) used in Mathematics Teaching Cycle, i.e. the "Hypothetical Learning Trajectory" (HLT).

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process–a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (Simon, 1995: 136)

The distinction between *hypothetical learning trajectories* and *local instruction theories* is that a hypothetical learning trajectory refers to a plan for teachers and researchers of instructional activities to conduct a particular teaching experiment in a given classroom on a daily basis. A local instruction theory, on the other hand, refers to a description of, and rationale for, the envisioned learning as it relates to a set of instructional activities for a specific topic, in this case addition up to 100 (Gravemeijer, 2004). In addition to this, Gravemeijer (2004) stated that a local instruction theory as a framework of reference could enhance the quality of the learning trajectories.

Bakker (2004) mentioned that the HLT has different functions depending on the phase of the design research and will continually develop and change through the different phases:

- During the design phase, the HLT guides the design of instructional materials that have to be developed or adapted.
- During the teaching experiment, the HLT functions as a guideline for the teacher and researcher for what to focus on in teaching, interviewing, and observing. It can be adjusted because of incidents in the classroom, such as anticipations that have not come true, strategies that have not been foreseen, activities that were too difficult, and so on.
- During the retrospective analysis, the HLT is compared to the actual learning process and it forms the basis for developing an instruction theory.

More specifically, this chapter clarifies a hypothetical learning trajectory preceded by a conjectured local instruction theory in addition up to 100, with the latter serving as a framework of reference in constituting the HLT.

4.1. The Conjectured Local Instruction Theory

A conjectured local instruction theory is made up of three components: (1) mathematical learning goals for students; (2) planned instructional activities and the tools that will be used; and (3) a conjectured learning process in which one anticipates how students' thinking and understanding could evolve while engaging in the proposed instructional activities (Gravemeijer, 2004). This design research will look at how Indonesian children who already use the algorithm strategy solve addition problems up to 100 using flexible mental arithmetic strategies.

The objective of the mathematics elementary school curriculum in number domain for the second graders in Indonesia is that children can solve addition and subtraction problems up to 500. However, calculation up to one hundred is important to lay the foundation for all further calculation with whole numbers, decimals, fractions, ratios, and percentages, because further calculation not only depends on the knowledge of number structure and the basic skill of flexible calculation, but also on the insight into fundamental calculation strategies (Treffers & Buys, 2001). Based on interviews with the teacher of the second graders and the second graders in SDN Percontohan IKIP Jakarta, I conclude that most of the children only know two strategies for solving such problems, the so-called "short strategy" and "long strategy", which have been taught for numbers up to 100 since they were in the first grade. The short strategy refers to the algorithm strategy or column arithmetic, while the long strategy is performing the algorithm strategy preceded by splitting numbers first.

"short strategy"	"long strategy"
1 47 68 + ——— 115	47 = 40 + 7 68 = 60 + 8 + ——— = 100 + 15 = 115

Figure 4.1. the 'short' strategy and the 'long' strategy

Observations before the teaching experiment show that children have inadequate conceptual understanding of numbers while solving problems using the algorithm strategy. For instance, Syifaa, an eight year old child, performed the algorithm strategy in solving an addition problem and treated numbers as concatenated single-digits (Fuson *et al*, 1997) without their values which cause a mistake.

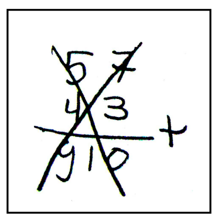


Figure 4.2. Syifa's mistake in solving an addition problem using algorithm strategy

This shows us that the second grade children in SDN Percontohan IKIP Jakarta are used to perform the algorithm strategy in solving addition problems. Children often do written algorithms without a great deal of thought, simply applying the procedure with very little sense of what they are really doing. Children not only make mistakes in treating numbers, but they also become nonflexible problem solvers in dealing with numbers in everyday life situations. The fact that the children cannot come up with strategies other than algorithm is also evidence that children are not flexible in using more efficient strategies to solve certain problems. Therefore, mental arithmetic strategies are suggested as an alternative to help them become more flexible problems solvers, and at the same time build their conceptual understanding of numbers and number relations (Treffers, 1991). However, this does not mean that algorithm is forbidden at all. Beishuzen (1993) suggested that algorithm is better taught after mental arithmetic strategies, to build on children's informal strategies and to minimize calculation errors because of the prematurely taught of algorithm

Treffers (1991) introduced the use of an empty number line in the realistic approach to describe the solution of (two digit) addition and subtraction problems by marking the numbers involved and drawing the jumps that correspond with the partial calculation. Moreover, Gravemeijer (1994a, 2000) pointed out three main reasons to choose an empty number line as a model. The first is based on the need for linear representation for counting numbers (Freudenthal, 1973), because the set (collected)-model of numbers cannot cover the need for a linear (sequence)-model for some situations such as distances, ages, and page numbers. The second reason is that the empty number line represents children's informal strategies such as counting on and counting down. The third is its level raising qualities that could foster the development of more sophisticated strategies. Regarding the level raising qualities, the empty number line allows children to express and communicate their own solution strategies, facilitating a classroom discussion on comparing more sophisticated strategies.

Aiming at an understanding of children's learning process in grade 2, as they construct mental arithmetic strategies on an empty number line to solve addition problems up to 100 through a measurement context, the ***mathematical learning goal*** below is formulated:

> *Children will be flexible in solving addition problems up to 100 both in context and in a bare number format using mental arithmetic strategies on an empty number line.*

Based on the fact that the children are already used to perform the algorithm strategy in solving addition problems, a learning trajectory must begin with a situation that differs from their previous experiences with numbers to avoid algorithm and to prompt new strategies. A measurement situation using a string of beads has been proved as a rich contextual situation to develop children's counting strategies, leading to the emergence from the situation of an empty number line into a model for calculating using mental arithmetic strategies (Gravemeijer, Bowers, Stephan, 2003). The teacher plays a crucial role in introducing the contextual situation and building the needs and reasons for children to involve themselves in the activity.

However, we must keep in mind that the empty number line is not a tool for doing calculation on paper. The role of the empty number line is intended as a model to record children's thinking process while doing mental calculation and at the same time to enhance more sophisticated strategies. Therefore, later on an empty number line is no longer used to record one's strategy in doing mental calculation.

It was observed before the teaching experiment that most of the children can count by tens, which will lead to make their counting more efficient. Therefore, children are expected to come up with a string of beads with 10-10 pattern. Facilitating discussions and posing questions are another crucial role of the teacher to help children who are still not able to count by tens. From this point, an empty number line occurs as a model of a measuring situation in which children record their measurement result. Finally, the empty number line becomes a model for doing mental strategies in solving addition problems flexibly.

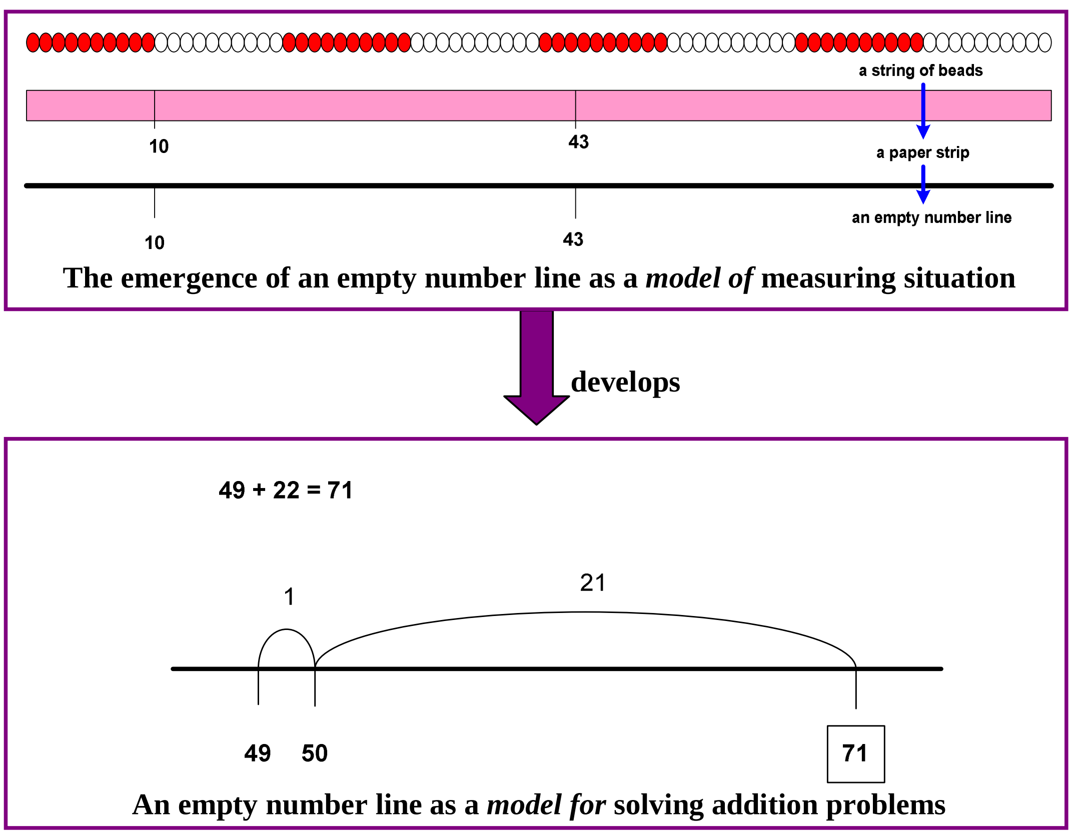


Figure 4.3. The emergence of an empty number line

In between the emergence of an empty number line from model of to model for, it is worth exploring number relations to enhance children's thinking in doing mental calculation flexibly. Doing activities in a playful manner, such as using number cards and games will attract children to engage in the activity. Combinations that make ten (1 and 9, 2 and 8, etc) give an idea of doing mental strategies using 'landmark numbers'. Number relations such as 38=30+8; 38+2=40; 38-3=35; etc will help children to make a proper decision in doing mental calculation for solving certain problems.

A contextual addition problem in the measuring situation could then be posed to children to explore their addition strategies. If none of them comes up with different strategies than algorithm, a teacher could use a counting by tens game while the jumps are represented on an empty number line to give a visual representation for their counting. Problem strings[1] on certain mental strategies could help children understand number relations while doing mental calculation.

The landscape of learning below illustrates for us the big ideas involving measurement and numbers, mental arithmetic strategies in solving double-digit addition problems, and the empty number line as a model for solving addition problems up to 100.

The Landscape of Learning
Big Ideas
Distance is measured as a series of iterated units. Numbers can be decomposed and the subunits or smaller amounts can be added in varying orders, yet still be equivalent (associative and commutative). There are place value patterns that occur when adding on groups of ten. Unitizing
Strategies
Counting three times Counting on Using the five and ten structures Keeping one number whole, using landmark numbers and or taking leaps of ten Splitting
Model
an empty number line

Figure 4.4. The Landscape of Learning (Catherine Twomey Fosnot, 2007: 5)

[1] A problem string is a structured series of problems that are related in such a way as to develop and highlight number relationships and operations (Fosnot & Dolk, 2001: 127)

- *Counting three times* is the lowest level strategy that the second grade students could do. To solve 38 + 27 for example, students count 1, 2, 3, …38 and 1, 2, 3, …27 then they count again 1, 2, 3…, 38, 39(1), 40(2), …65(27). This strategy takes a lot of time and is not efficient for bigger numbers.
- *Counting on one by one* is the next level after students can count three times, this strategy is more efficient than counting three times, but it is still not efficient for bigger number. To solve 38 + 27, students start counting from 38, then they count on 39(1), 40(2), …65(27).
- *Counting on using the five and ten structures* is more efficient than counting on and backward one by one.
- *Jumps of tens* strategy treats one number as a whole, while the second number is added by taking leaps of tens
- *Jumps via tens* strategy treats one number as a whole, while the second number is added via landmark numbers (10, 20, 30, etc)
- *Splitting* strategy involves splitting the tens and ones and addressing these separately.

4.2. The Hypothetical Learning Trajectory

4.2.1 *Measuring with a string of beads*

Mathematical Learning Goals:

1. Children can measure things around them using a string of beads.
2. Children can use certain patterns on the string of beads to make their counting easier.
3. Children can compare numbers in the measurement context.
4. Children can count by tens in determining the number of beads with the 10-10 pattern.

Tools:

Multiple color of beads , strings, children's worksheets

Planned Instructional Activities:

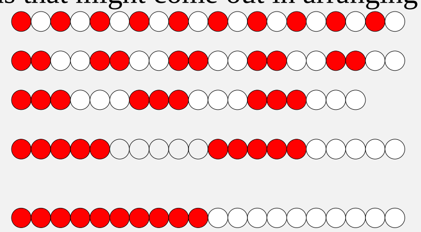
A. Developing the context
   - Beads are for accessories in everyday life.
   - How many beads do you need to make a bracelet for yourself? What about your friends? Who need the biggest number of beads or the smallest number of beads? Why is the number of beads used in your bracelet different?
   - An idea of measuring using a string of beads occurs after the discussion of different number of beads that the children need to make bracelets.

B. Measuring for Celebrating the 63rd Indonesian Independence day
   - Children work in pairs.
   - Children measure the circumference of their heads to produce a paper headband to celebrate Indonesian Independence Day.
   - Children have freedom to measure things around them.
   - While the children are measuring, the teacher and researcher observe the children's strategies in counting and determining their measurement results.

C. Math congress [2]
The classroom community discusses and compares the structure that they use in their string of beads. They argue which structure is helpful for their counting, that is to speed up their counting.

Hypotheses of Children's Learning Process:

These are some patterns that might come out in arranging the beads:



The children might or might not apply the structures to help them counting. Some children still count by ones, while others can count by twos, fives, or tens.

Interrupting children when they are counting the number of beads can frustrate them when they are counting one by one, while the ten structure in structuring the beads can help children in determining the result of measurement easily by doing counting by tens. Every group has a worksheet to be filled in with the results of measuring things in the classroom.

Later, the classroom community discusses which structure of bead strings is best to use in measuring. The children who found difficulties and took a long time to determine the result of their measurement are expected to realize that the ten-structure is helpful in determining the number of beads easier. However, there might also be children who prefer the five-structure instead of ten. Bring a problem to children in which they have to show a number of beads and let them judge which structures help them find the fastest solution. After the classroom community comes to an agreement about the ten-structure, a string of 100 beads can be put on the blackboard.

It is expected that children are already well acquainted with the number sequence up to 100 before doing this activity, or on the other hand this activity might also encourage children to be able to recite the number sequence up to 100.

4.2.2 *The emergence of an empty number line*

Mathematical Learning Goals:

1. Children can measure using paper-string of beads with 10-10 pattern.
2. Children can represent measurement results on an empty number line.
3. Children can find a difference between two measurement results using the paper-string of beads, or using an empty number line

[2] A math congress is a whole-group share where the children communicate their ideas, solutions, problems, proofs, and conjectures with one another. Moreover they also defend their thinking (Fosnot & Dolk, 2001: 27)

<u>Tools:</u>

String of beads with 10-10 pattern; paper-string of beads with 10-10 pattern; children's worksheet; number cards in big size (1 – 100)

<u>Planned Instructional Activities:</u>

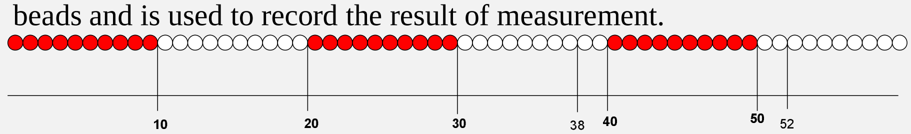
A. Class Discussion
- ❖ Measuring on paper-string of beads (10-10 pattern)
- ❖ The emergence of an empty number line: record the measurement results on an empty number line
- ❖ Finding a difference between two measurement results

B. Working on the worksheet

<u>Hypotheses of Children's Learning Process:</u>

The paper strip (with circles as a representation of the beads) below the string of beads is intended to let children understand the modeling situation and moreover to emphasize the 10-10 pattern that could enhance children's counting strategies. Children are expected to be able to measure things using this 10-10 paper-string of beads and recognize the 10-10 pattern to count by tens.

A line without numbers is placed under the paper-string of beads and is used to record the result of measurement.



Recording the measurement result on the empty number line is an activity to locate numbers in an empty number line. The string of beads or the paper strip is put above the empty number line, so that children can still count by ones to locate a number on an empty number line. Children may do counting by ones or by tens, but once a child does counting by tens, there is an impression that this is a very sophisticated way to locate numbers in an empty number line. The ten-structure in the bead string itself encourages children to count by tens instead by ones or by fives. When recording the tens in the empty number line, make sure that the numbers are written bigger than other numbers, it will help children to recognize the ten number sequence (10, 20, 30, …, 100).

After some numbers are recorded on the empty number line, ask the children to find the difference or the sum. Some questions that could be posed are:

"*Now we know that the length of the table is 58 beads and the height of the table is 47. Could you find out the difference between the two measurements?*"

"*Can you explain to us how you find the difference between 58 and 47?*"

In this problem, children might solve this problem by adding on the number 47 until it comes to 58. They probably do counting by ones (48, 49, 50,… 57, 58) with their fingers, or adding 10 to 47 to get 57 and add 1 more, or probably doing algorithm (column arithmetic). By providing the paper strip above the empty number line, children can count the beads between two numbers on the paper strip to find their difference. This will give an idea to solve subtraction problems using adding on strategy.

4.2.3 *Exploring number relations*

Mathematical Learning Goals:

1. Children can memorize combinations that make ten in a playful manner.
2. Children can estimate a number position on an almost empty and empty number line.
3. Children can locate numbers on an empty number line using number relations.

Tools:

Paper-string of beads with 10-10 pattern, children's worksheet, number cards in big size (1 – 100), ropes / strings

Planned Instructional Activities:

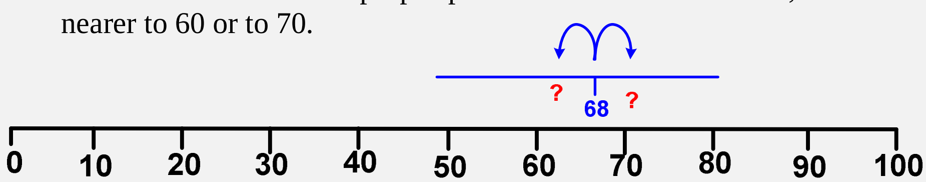
A. A game on combinations that make ten
B. Locating numbers on an almost empty number line
C. Locating numbers on an empty number line
D. Exploring number relations

Hypotheses of Children's Learning Process:

Children remember combinations that make 10 by playing a game. This will help them to be more flexible in solving addition problems, for instance using strategy "jumps via tens".

In locating numbers or number cards using the bead string, children might still count ones. Locating number cards on a string under the paper strip in a playful manner will give a chance to children who still count by ones to develop their strategies through discussion

Afterwards, an almost empty number line (a number line with a sequence of tens on it) is introduced as a model of the 10-10 pattern string of beads. Locating numbers on an almost empty number line without help of the paper strip can urge children to estimate the distance between two numbers. They might still count one by one for small numbers, but problems occur when the numbers are bigger, and they must use number relations to locate a number in its proper position. For example, locating 68 on the almost empty number line is quite difficult when children count by ones from the beginning. The easier way is for children to estimate the proper position between 60 and 70, whether it is nearer to 60 or to 70.



Locating numbers on an empty number line for the sequence of tens will help children recognize the sequence of tens and urge them to use number relations (will 60 be on the right or the left side of 50? is 48 closer to 40 or 50? how many more to get 50 from 48? etc).

Having a framework of number relations, children will be more flexible in solving addition and subtraction up to 100 using mental arithmetic strategies. An example is the jumps *via* tens strategy employing the number relation: 38 + 57 = **38 + 2** + 55 = 40 + 50 + 5 = 95

4.2.4 *Exploring addition strategies on an empty number line*

Mathematical Learning Goals:

1. Children can use the strategy 'jumps *of* ten' in solving addition problems on an empty number line.
2. Children can use the strategy 'jumps *via* ten' in solving addition problems on an empty number line.
3. Children can use both strategies using a compensation technique in solving addition problems on an empty number line.

Tools:

Children's worksheets

Planned Instructional Activities:

A. A game on jumps of ten
B. Solving addition problems up to 100 using jumps of ten strategies
C. Working on the worksheet

Hypotheses of Children's Learning Process:

Children are expected to come up with different strategies, such as counting by tens, using landmark numbers, splitting, etc in solving addition problems. Afterwards the teacher could represent their thinking on an empty number line.

A game in counting by tens while the teacher represents their jumps of ten on an empty number line will let children see that every time they add ten to the number, a curve line (a jump) is drawn on the empty number line. This will give an idea of the strategy "jumps of tens" to do addition.

If they do not come up with one of the strategies above, the teacher could give a hint to start with the strategy counting by tens (jumping by tens) with the game above as the starting point to introduce the strategy on an empty number line.

A string of problems (pose to the children once in a time), such as: 75 + 10; 75 + 14, or 62 + 30; 62 + 32, will help children to be more competent in strategy "jumps of ten". Children are expected to know by heart every jump of ten, that is 75+10 = 85, then they are expected to use this result to find out the result of the next problem (if they realize that the next problem has 4 more for the second number than the first problem), that is adding 4 to the previous result (85) becomes 89.

The problem string is expected to help children practice their mental strategies and apply their number relations in solving the problem.

4.2.5 *Developing addition strategies on an empty number line*

Mathematical Learning Goals:

1. Children can solve addition problems up to 100 using the 'jumps of tens' strategy on an empty number line.
2. Children can solve addition problems up to 100 using the 'jumps via tens' strategy on an empty number line.
3. Children can solve addition problems up to 100 using both strategies above by applying the 'compensation' technique.
4. Children will be flexible in solving addition problems up to 100 on an empty number line.

Tools:

Children's worksheets

Planned Instructional Activities:

A. Discussing the 'jumps *of* tens' strategy in solving addition problems up to 100.
B. Posing 'problem strings', for example: 72+10; 72+12; 75+12; etc
C. Developing the 'jumps *via* tens' strategy
D. Games
E. Comparing more efficient strategies for certain addition problems.

Hypotheses of Children's Learning Process:

In the discussion, children show their performance in solving an addition problem using different strategies. For the 'jump of ten' strategy, children might come up with a bigger jump, such as 20, 30, or 40. This will encourage the children's level of thinking through discussion on the more efficient strategies. If some children come up with the strategy 'jumps via tens', then it could be a topic of discussion in the classroom to compare more efficient strategies for certain types of addition problems.

A game and context problems are posed to children to introduce subtraction problems like adding on. Children can compare which strategies are best for certain problems. They can decide by themselves and use an empty number line to represent their thinking.

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The HLT has elaborated the conjectured local instruction theory in which the RME principles underpin the whole instructional sequence. The HLT is expected to be able to support children's progressive mathematization from the reality (measuring activity) to the formal mathematics (addition up to 100 using mental arithmetic strategies on an empty number line). Together with the guidance of the teacher and the open classroom culture, these are expected to support the development of children's learning process to be a flexible problem solver.

5. Retrospective Analysis

The retrospective analysis in this design research encompasses the explanation of data both in general and in specific cases. The learning process development of a number of children will be analyzed, not only for the children as individuals, but also for their participation in and contribution to the development of classroom mathematical practice. The role of the teacher and the classroom culture are highlighted as well. On top of that, the hypothetical learning trajectory is compared to the actual learning process that happens in the teaching experiment. Results of the retrospective analysis will form the basis for adjusting the new HLT and for answering the research questions. The following section describes the interpretative framework that guides the retrospective analysis of children's learning process in constructing mental arithmetic strategies in a classroom community.

5.1. Interpretative Framework

As we have noted before, this design research which attempts the development of a local instruction theory has a prospective and reflective characteristic in a cyclically iterative process (Cobb *et al*, 2003). On the prospective side, we initially conduct an anticipatory thought experiment by envisioning how mathematical activities and discourses may evolve as proposed types of instructional activities are enacted in the classroom, thereby developing conjectures about both possible learning trajectories and the means that might support that learning. On the reflective side, the tested instructional activities together with the conjectures are refined and generated as informed by ongoing analyses of both the students' reasoning and the development of classroom mathematical practices.

In this section, we describe a methodology for analyzing the collective learning of a classroom community in terms of the evolution of classroom mathematical practices. Cobb *et al* (2001) describe criteria for an appropriate analytical approach:

1. It should enable us to document the collective mathematical development of the classroom community over the extended periods of time covered by instructional sequences

2. It should enable us to document the developing mathematical reasoning of individual students as they participate in the practices of the classroom community
3. It should result in analyses that provide feedback to inform the improvement of our instructional designs.

In this respect, the interpretative framework below (figure 5.1) – that coordinates a social perspective on communal practices with a psychological perspective on individual students' diverse ways of reasoning as they participate in those practices – is proposed to organize our analyses of classroom mathematical practices. Analyses of this type are appropriate for the purposes of developmental research, because they document instructional sequences as they are realized in interaction in the classroom (Cobb & Yackel, 1996).

Social Perspective	**Psychological Perspective**
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Socio-mathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical interpretations and reasoning

Table 5.1. An interpretative framework for analyzing communal and individual mathematical activity and learning (Cobb, 2001)

The reflexive relation between the social perspective and psychological perspective (Cobb & Yackel, 1996; Cobb *et* al, 2001) in each row is described below:

Classroom social norms characterize regularities in communal classroom activity and are considered to be jointly established by the teacher and students as members of the classroom community. Examples of social norms include explaining, justifying solutions, attempting to make sense of explanations given by others, indicating agreement or disagreement, and questioning alternatives when a conflict in interpretations becomes apparent. The teacher is seen to express an authority in the classroom by initiating, guiding, and organizing the renegotiation process. However, the students are also seen to play their part in contributing to the evolution of social norms. In this joint perspective, classroom social norms are seen to evolve as students reorganize their beliefs, and, conversely, the reorganization of these beliefs is seen to be enabled and constrained by evolving social norms.

In *socio-mathematical norms,* we limit our focus in subsequent analyses to the normative aspects of whole-class discussions that are specific to students' mathematical activity. Examples of such socio-mathematical norms include what counts as a different mathematical solution, and an acceptable mathematical explanation. The teacher guides the development of a community of validators in the classroom such that claims were established by means of mathematical argumentation rather than by appealing to an authority such as that teacher or a textbook. The students could themselves judge what counted as a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation. However, these were precisely the types of judgments that are negotiated when establishing socio-mathematical norms. Therefore, students develop specifically mathematical beliefs and values that enable them to act as increasingly autonomous members of the classroom mathematical community as they participate in the negotiation of socio-mathematical norms.

Classroom mathematical practices – A conjectured local instruction theory can be viewed as consisting of an envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practice. In analyzing the evolution of classroom mathematical practices, we can document the actual learning trajectory of the classroom community as it is realized in interaction. In identifying sequences of such practices, the analysis documents the evolving social situations in which students participate and learn. Students actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities and, conversely, these reorganizations are enabled and constrained by the students' participation in the mathematical practices.

In summary, the psychological perspective focuses on students' diverse ways of reasoning while participating in mathematical practices, while the social perspective brings out the development of mathematical practices as a result of the social interaction among the members of a classroom community. The distinctions between the three aspects under the social perspective is that the social norms include what counts as explaining and justifying interpretations, while the socio-mathematical norms are more specific to mathematical activity in which the classroom community come to an agreement of what counts as mathematical argumentation and justification. In contrast to socio-mathematical norms that are specific to mathematical activity, classroom mathematical practices focus on the *taken-as-shared*[1] ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas (Cobb *et al*, 2001).

5.2. The Comparison between the HLT and the Actual Learning Process

The actual learning process was documented in a series of video-recorded classroom situations conducted in the first three weeks of August. Before the HLT was tested in the classroom environment, some information was gathered by conducting an interview with the teacher with whom we were going to collaborate; interviews with the second graders; observations of classroom situations; trying out activities; and a new classroom culture was introduced to children (see appendix 4) using a poster with pictures and messages on it. The interviews and observations provided essential information about the current classroom culture and the role of the teacher in the classroom. Interviews with children and the try out activities support my hypotheses about children's counting and calculation strategies in dealing with addition problems up to 100 and contribute to the development of the HLT (see appendix 5).

The following section is the general description of the development of classroom mathematical practices in the teaching experiment compared to the hypothetical learning trajectory (HLT). Further analyses of the data are expounded in the next section where the emergence of the classroom mathematical practices is analyzed in coordination with the individual learning process.

5.2.1. Measuring with a String of Beads

If we look back to the description of the HLT in chapter 4, there were some patterns that were expected to come out when children were measuring, such as 1-1 (alternating colors every one bead), 2-2 (alternating colors every 2 beads), 5-5 (alternating colors every 5 beads), 10-10 (alternating colors every ten beads), etc. Some children might use

[1] Taken as shared: We speak of normative activities being taken as shared rather than shared to leave room for the diversity in individual students' way of participating in these activities. The assertion that a particular activity is taken as shared makes no deterministic claims about the reasoning of the participating students, least of all that their reasoning is identical (Cobb *et al,* 2001)

this pattern to help them counting but some might not. In fact, none of the children come up with the 10-10 pattern of the string of beads. Most of them have 1-1 or 2-2 patterns or two different colors on both sides (see the analysis in the section ***5.3.1(2)***).

Children were also expected to be able to do measurement using a string of beads and realize that they are counting the number of beads covering the span of the measured things. Although children seem to have no difficulties with the measurement activity, a closer look on how children interpret measurement gives a detailed analysis of their conceptions of measuring (see the analysis in section ***5.3.1(1)***).

The math congress was intended as a forum to share children's strategies in counting the number of beads and comparing the effective counting strategies using patterns. By providing different patterns of string of beads, children were expected to come to counting by tens or by fives strategies. The hypothesis is proven that at the end, a child count by using the ten structure after the teacher supports her thinking by posing questions and providing tools (see the analysis in section ***5.3.1(3)***).

5.2.2. The Emergence of an Empty Number Line

As expected, children can do measurement using a paper-string of beads (a paper strip with circles as a representation of the beads). Although there is one child who still counts by ones, many of them already count by tens. Giving long objects to measure encourages children to count by tens using the 10-10 pattern. The ten-structure in the paper-string of beads above the empty number line encourages children's counting strategies as it was expected before. It seems that children do not have any problems when recording the results of measurement on an empty number line under the paper-string of beads. Most of them already can count by tens in locating numbers on an empty number line under the paper-string of beads. The empty number line has emerged as a model *of* measuring situation (see the analysis in section ***5.3.2***).

5.2.3. Exploring Number Relations

The proposed activity on exploring number relations is locating number cards on a string followed by locating numbers on an empty number line. Before locating numbers activity, the teacher proposes a game of combinations that make ten and jumps of ten. As expected before, almost all the children can mention combinations that make tens and play the jumps of tens game (see analysis in the section ***5.3.3(1)***).

Locating number cards on a string is performed with the help of the paper-string of beads above the string. Children still use the paper strip to locate numbers, either counting one by one or counting by tens. When bigger numbers are given, children are challenged to count by tens. Some of them only need to look at the pattern of the paper strip without pointing while counting by tens (for smaller numbers) and some of them using number relations to locate numbers. The discussion of comparing strategies in locating numbers develops children's number relations (see the analysis in section ***5.3.3(2)***).

When the paper strip is removed from the white board, some children still count one by one and imagine that there are beads on an empty number line, but others already count by tens. The activity of locating numbers on an empty number line without paper strip helps children to develop their number relations as expected (see analysis in section ***5.3.3(3)***). The children use their number sense and number relations in locating numbers between 0 and 100 on an empty number line.

5.2.4. Exploring Addition Strategies on an Empty Number Line

In the HLT, children were expected to come up with different strategies, such as counting by tens, using landmark numbers, splitting, etc in solving addition problems. Afterwards the teacher could represent their thinking on an empty number line. In fact, there is a girl named Maudy who comes up with the strategy of jumps of tens in solving 33 + 25, and then the teacher uses an empty number line as a model for representing Maudy's strategy (see the analysis in section ***5.3.4(1)***). Once a child comes up with the jumps of tens strategy, all children use the same way in solving addition problems in their book.

When another problem is posed on the whiteboard, the teacher challenges children to use different strategies. The more strategies they could find, the more appreciative the teacher was. Children show their performance by doing calculation on an empty number line in different strategies. The string problems are not given one at a time, but all at once. Therefore, there is no discussion and sharing of more effective strategies in solving a certain problem, and the children tend to work individually on their paper and try to solve problems like the example before. However, the classroom discussion reveals different addition strategies on an empty number line, including the 'jump-via-ten' strategy (see the analysis in section ***5.3.4(2)***).

5.2.5. Developing Addition Strategies on an Empty Number Line

In the discussion, children show their performance in solving an addition problem using different strategies. There is an expectation that during the classroom discussion, children will come up with a bigger jump, such as 20, 30, or 40 on an empty number line in solving an addition problem. The teacher's guidance and the representation of children's thinking on an empty number line facilitate the discussion that leads to the more efficient strategies in solving the problem (see the analysis in section ***5.3.5(1)***).

In the HLT, a game and context problems are posed to children to introduce subtraction problems as adding on. In the actual learning process, the teacher shows her authority by giving a direct instruction to children to use an empty number line to solve the missing add-end problem. Although children's performances in front of the classroom seem okay, they do not have sufficient understanding in dealing with the problem (see the analysis in section ***5.3.5(2)***).

5.3. Data Analysis

The primary aim when presenting an analysis is on the evolution of classroom mathematical practices. Therefore, our unit analysis is that of a classroom's mathematical practice and students' diverse ways of participating in and contributing to its constitution. It should be noted that in making reference to both communal practices and individual students' reasoning, this unit captures the reflexive relation between the social and psychological perspectives on mathematical activity.

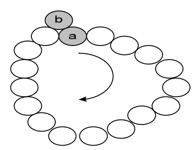
The episodes below were selected to clarify the development of mathematical practices aiming at flexibility in doing mental arithmetic strategies on an empty number line.

5.3.1. Measuring with a String of Beads

<u>Data Descriptions and Interpretations</u>

1) *Measuring as covering distance*: Rizqy and Fahri's performance in measuring activity

Me : What are you measuring?
Fahri : My head
Me : How do you measure that?
Rizqy : By counting
Me : Where do you have to start and stop in the counting?
Fahri : From one to any number
Rizqy : Starting from this (pointing to the bead 'a' in the picture – as an example) and ending at this (pointing to the bead 'b')



Me : Ok, now count the beads!
Rizqy : One, two, three,…thirty seven…(he is distracted by Fahri and loses his counting). Yaaaa…I have to start from the beginning. One, two, three, …, forty one (while tagging the beads one by one)
Me : Do you think it is a long way to count or not?

….

Fahri: : Ten(pointing to the first group of blue beads using his palm), twenty(pointing to the second group of white beads using his palm)
Rizqy : You are wrong!
Me : So, where is the ten?
Rizqy : (looking at the beads without pointing to them) from this (pointing to the first bead) until this (pointing to the tenth bead)
Me : Then?
Rizqy : (looking at the beads again without pointing to them) from this (pointing to the 11^{th} bead) until this (pointing to the 20^{th} bead)
Me : So, what is the length from this (the 1^{st} bead) to this (the 20^{th} bead)
Rizqy : It's twenty, twenty beads.
Me : Do you think you need to change the pattern of your string to make your counting easier?
Rizqy : I don't know

I observed that Rizqy has inadequate understanding of the measurement concept. It happens probably because he cannot immediately see the span in a curve line when he is measuring a round thing. Therefore, he points to the incorrect end bead in his measuring activity. However, Rizqy has a sophisticated way of determining the sequence of ten beads by only looking at the string of beads without tagging. On the other hand, Fakhri points to the first group of blue beads as ten beads, and the next group of the red beads as another ten beads, although it is obvious that the alternating groups of blue and red beads have different lengths. This shows Fakhri's lack of number sense, because he can not perceive that the length of the string of beads is comparable with the number of beads in it, i.e. the longer the string of beads, the bigger the number of the beads.

Children in the classroom are given the choice to count things around them, but since the context is developed from measuring the wrist, almost all of them measure parts of the body (almost all round things). From my observation, children can't come up with the 10-10 pattern because the string of beads is used mainly to measure round things, so that they can't immediately recognize that patterns while measuring that could help them in counting. This might explain the answer of Rizqy "I don't know" when he is asked about changing the pattern of the string of beads.

2) The need of measuring in arranging beads on a string

The teacher distributes a pack of colorful beads to every pair of children, right after the classroom community discussion about classroom culture. The teacher then immediately asks children to arrange the beads on a string that has two end points. Afterwards, children start to open the packs of beads and they arrange the beads in the string. However, the teacher gives the children the choice to have different patterns on their string of beads.

Most children have the patterns 1-1 or 2-2, and none of them comes up with the pattern 10-10 (illustrated below)

Pattern 1-1: ●○

Pattern 2-2: ●●○○●●○○●●○○●●○○●●○○●●○○●●○○●●○○●●○○●●○○

Pattern 10-10: ●●●●●●●●●●○○○○○○○○○○●●●●●●●●●●○○○○○○○○○○

After all the children are ready with their string of beads, the teacher starts a discussion in the classroom:

T: Now, show your string of beads
S: (show their string of beads by holding it up)
T: What is this string of beads for?
S: Necklace (shout together)
T: What else?
S: Bracelet (shout together)

None of the children answer that the string of beads could be used as a tool in measuring, and none of them come up with the 10-10 pattern. This might happen because the teacher does not build a need for measurement into developing the context. We saw in the description above that the teacher immediately asks the children to arrange the beads on the string without giving any mathematical reasons to do so, and the obedient children just do what the teacher asks.

The teacher is seen to express her dominant authority in the classroom by initiating and guiding the mathematical activity in measuring. This shows a common social norm in most Indonesian classroom that the children often are not provided with a mathematical reason in engaging a mathematical activity. This might cause the mathematical activity become meaningless in children's thinking.

3) ***Enhancing children's thinking to count by tens by posing questions and providing tools***

The teacher put different strings of beads with different patterns on the whiteboard. Maudy is asked to find out the number of beads for each string of beads in front of the classroom. The first string has a 1-1 pattern, and Maudy counts one by one. Then the teacher asks Maudy again to find out the number of beads on the second string which has 2-2 pattern, and Maudy keeps counting one by one; until she is asked to count the number of beads from the longest string which has a 10-10 pattern. Then she says "aaaaaahhh". Then she starts thinking and counts the first 10 beads one by one. Once she recognizes the pattern of the string of beads, she counts the number of beads by tens.

By providing different patterns of the string of beads and posing challenging questions, the teacher could help children to improve their counting strategy. The 10-10 pattern of the string of beads could encourage children to count by tens.

<u>Answers to Research Questions and Recommendations</u>

The data description and interpretation in 5.3.1(3) can answer question (2.2) and question (3) at the same time. The pattern of the string of beads supports the development of children's counting strategy from counting by ones to counting by tens. Alternating the color by tens in a string of beads allows children to see the beads as chunks of ten beads, which leads to counting by tens. Moreover, the teacher plays her role by providing tools (several strings of beads with different patterns) and posing questions to enhance children's thinking. In addition to this, the teacher often repeats children's strategies which give other children a chance to understand the strategy.

The data in 5.3.1(1) and 5.3.1(2) gives an answer to question (2.1) and question (3). The proposed measuring activity does not help children to enhance their calculating strategies and to come up with the intended patterns. The context of measuring children's wrists to make bracelets gives the children the idea to measure all round things on their body, such as the circumference of their waists and necks. As a result of these round things, children cannot immediately perceive the pattern of a string of beads, which could improve their counting strategies. This suggests to us that a next time, it will be better to start the measuring activity with measuring straight lengths. Moreover, the lack of a mathematical reason for measuring in arranging the beads might also contribute to the unsuccessful activity where none of the children come up with the expected patterns. In this case, the teacher's role in initiating and guiding the development of measuring context in the classroom also takes part.

On top of that, to make children really experience the need of using patterns in measuring, the measuring activity should be repeated once more after a classroom discussion in which children compare different patterns and come to the conclusion that using a certain pattern can help them counting. This will make all children get involved in experiencing counting the number of beads by tens (or by fives).

5.3.2. The Emergence of an Empty Number Line

<u>Data Descriptions and Interpretations</u>

Alfi is asked to measure a length of a plastic ruler using a paper-string of beads on the blackboard. Alfi still counts by ones in determining the length of the ruler, she gets 11 as the result. After measuring, she is asked to record her measurement result on an empty number line under the paper-string of beads. In recording the result, Alfi simply connects the paper-string of beads (at the end of the 11th beads) to the empty number line by using the rules and write 11 on the empty number line.

The second children, Rafita measures the length of a rope. Rafita can count by tens in determining the measurement result. She does not need to tag or count the beads out loud. She just looks at the pattern and simply records her measurement result.

The empty number line emerges very naturally as a model of the measurement situation. Children record their measurement result by connecting the beads on the string of beads to the line under it. Although some of them still do not use the pattern of ten in counting, providing a long object to be measured can urge children to count by tens.

Answers to Research Questions and Recommendations

The empty number line emerges as a result of the measurement situation through the activity of recording a measurement result. This fact contributes for an answer for the second question about the emergence of an empty number line through the measurement context. The empty number line could be used to represent numbers in a line model. After the empty number line emerges as a model of the measurement situation, it will develop as a model for mathematical reasoning through the process of exploring number and its relations as mathematical objects. Moreover, this part of the analysis also answers the question (2.2) of whether the 10-10 pattern of the paper-string of beads encourages children's counting strategies. The pattern helps children to count by only looking at how many chunks of ten beads there are and how many more to the end of the measured object. This in fact is a very sophisticated strategy in counting mentally supported by the pattern.

5.3.3. Exploring Number Relations

Data Descriptions and Interpretations

1) ***Combinations that make ten and jumps of ten***

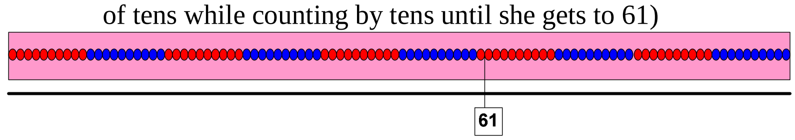
> In this game, the teacher mentions a number (from 1 to 9) and the child that is pointed at has to mention its pair to make ten. In this game, almost all the children can mention automatically and correctly. Only three out of 36 children have difficulties and haven't automatized combinations that make ten. In the game jumps of tens, the children seem to forget the classroom culture that has been approved before. Children shout together in answering the teacher's questions. Then the teacher reminds the children to answer only when the teacher points to one of them. After that, the children don't shout together again in answering the questions.

In the first game, the children answer the teacher's question in turn because the rules in the game mean that children can't shout together. In the second game, children shout together, because the teacher poses questions without pointing to one of the children. Therefore, the teacher reminds the children not to shout together because they have an agreement before that the children can only give answers when the teacher asks them to do so (see the classroom culture in appendix 4). The establishment of the classroom culture needs consistency from the teacher to always remind the children when they are breaking the rules, and also need maintenance from the classroom community.

2) *Locating number cards on a string under paper-string of beads with 10-10 pattern*

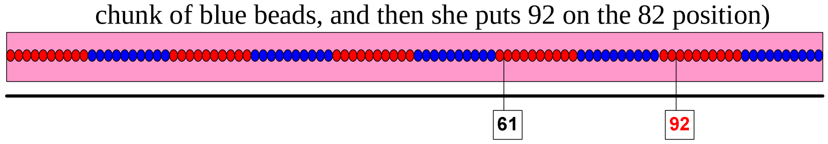
The teacher : "Who can put a number card for 61 on the string below the paper strip?"

Shafa : (Shafa locates 61 on 51 then another child say that it's wrong. Shafa counts once again carefully by moving her finger along the chunks of tens while counting by tens until she gets to 61)



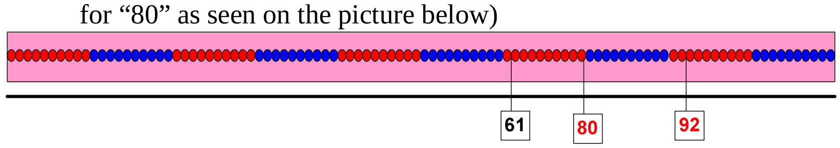
The teacher : (Gives the second turn to Meidy)

Meidy : (Meidy doesn't count from the beginning, she simply looks at the end of the paper-string of beads and points to the two red beads after a chunk of blue beads, and then she puts 92 on the 82 position)



No one pays attention to Meidy's mistake in locating numbers so the teacher moves to the next children. The teacher asks Rezon to put number 80.

Rezon : (He looks at his number and counts silently the sequence of tens starting from 60, "sixty, seventy, eighty". But then he seems confused and takes a while to rethink, then his mouth motion shows that he mentions "ninety two". Not long after that, he places the number card for "80" as seen on the picture below)



The interpretation of Rezon's thinking:

Based on his previous performance in locating the number card of 48 (*see the analysis (*) locating numbers on a string of beads*), I observed that Rezon counts on his understanding of number relations to locate numbers. In this activity, he makes a mistake in locating "80" exactly because of his point of reference. Rezon takes "92" as his point of reference in locating "80, therefore he thinks that two beads before 92 must be 90 and thus he puts 80 ten beads before 90. He doesn't check his answer from a different point of view, that is from 61, maybe because he knows that 80 is closer to 92 than to 61, so it will give him less time and effort to locate the number card of "80" from 92. However, he does have a good understanding of number relations, even if he has less self reflection on his answer

(*) Locating numbers on a string of beads (Rezon)

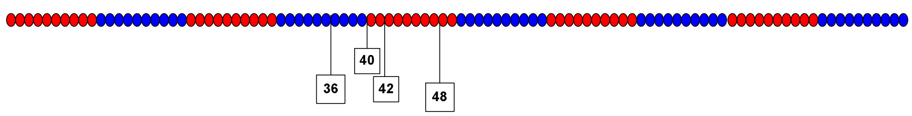
The activity was locating results of measurement from the previous day on the string of beads with the 10-10 pattern. There were number cards for 36, 40, and 42 on the string of beads before Rezon came to put the number card of 48

The teacher : "Rezon! What was your result of measurement yesterday?"

Rezon : "I forgot"

The teacher : "forgot…Ok, here come to the front"

Rezon : goes to the teacher and sees his measurement result on his worksheet, then starts to write the number on a card, then moves to the string of beads on the white board. He points to "42", the known number position and in less than 10 seconds he puts the number card for "48" in the right place without counting from the beginning (his hands / fingers don't show that he is counting and pointing)



The interpretation of Rezon's thinking:

His performance in dealing with this problem shows that he has a good understanding of number relations. I could argue from this fact that the activity of locating numbers on a string of beads could help children enhance their understanding of number relations as was conjectured before. Rezon only points to 42 as the nearest known number to 48 and probably he also looks at the position of 50 in his mind to be able to locate 48 without counting on from 42. If he starts from 42, then he needs to count on "43, 44, 45, 46, 47" and then come to 48 that requires keeping track of his counting using his finger. However, he does not use his fingers besides pointing to 42, so I assume that he knows that forty eight is two less than fifty.

The teacher: (asks the fourth child, i.e. Michelle to locate 30)

The children are very eager to contribute in locating numbers by raising their finger, hoping that the teacher will choose them.

Michelle: (She can count by tens from a distance without touching the paper strip. She uses her finger to point to the paper strip and then she knows where to put 30 correctly)

The next child pointed by the teacher is Dhea:
Dhea : (She can easily put 13 because she realizes that the first red chunk of ten is a group of ten beads, so she just add three more)

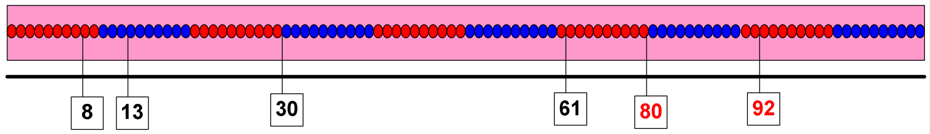
The next child is Fakhri:
Fahri : (He counts by one and tags the circles to locate 8 properly)

The teacher : "How does Fahri count in locating number 8?" (asking to all children)
The children : "He counts by ones!" (shout together)
The teacher : "Who has another strategy?"
Bita : " five plus three is eight"
The teacher : "Any other strategy?"
Yona : "Ten minus two is eight"

The teacher is trying to compare Fahri's strategy and Yona's strategy in locating number 8. Then Yona shows her strategy in front of the classroom that she moves two beads backwards from the tenth beads.

The teacher : "Can we use Yona's strategy?"
The children : "Yes, we can!"
The teacher : "Which one is faster, Yona or Fahri's strategy?"
The children : "Yona"

The pattern of the string of beads does help children in counting the number of beads. They can easily look at the pattern without counting the beads by ones or even tagging. The teacher facilitating the discussion by posing the question: "who has another strategy?" The question encourages children's thinking to find another sophisticated strategy to locate 8 on the number line without counting by ones. Therefore, the number relation i.e. eight is two less than ten, is developed by the classroom discussion. In this case, Yona shows her good understanding of number relations that is eight and two make ten, so eight can be obtained by taking away two from ten. Yona has given a significant contribution to the classroom community and the classroom community agrees that Yona's strategy is better than the others. Furthermore, the teacher's question - "Which one is faster, Yona or Fakhri's strategy?" - engages all the children in the process of coming to an agreement for the more effective strategy.



After the teacher confirms Fakhri's strategy, she continues to Dhea's strategy in locating 13. Dhea says that she counts by tens. Next, Michelle confirms that she count by tens. Then, Shafa confirms that she also counts by tens. Afterwards, the teacher asks for Rezon's strategy and the classroom community finds out that 80 and 92 are in the wrong place. The teacher asks Rezon to look back to his answer and correct his mistake. Rezon looks at the previous number (61) and use it (probably he uses 60) as the starting point to locate 80 in the right place. Then Meidy also manages her mistake. She looks at the beginning of the paper-string of beads and without pointing she knows where to put 92 correctly.

The teacher	: "Is there any other strategies in locating 92 besides counting by tens like Meidy? Jenna? "
Jenna	: "a hundred is subtracted by 8"
The teacher	: (repeat Jenna's answer)

By asking what strategies the children use in locating numbers, the teacher attempts to let the others understand how the children have come to an answer. Moreover, the question also makes the children's mistakes visible. The teacher builds the children's sense of responsibility by asking them to manage their own mistakes. Rezon realizes that starting from 92 and stepping back to locate 80 was a mistake, because 92 is not in the right place, therefore he changes his starting point from the other side, that is from 60, then he counts on 60, 70, 80. Rezon makes use of the known number 60 to go to 80. On the other hand, Meidy does not make use of the known number to go to 92. She counts by tens from the beginning of the string of beads.

Afterwards, Jenna who is one of the high level children in the classroom, including Yona, comes up with a sophisticated way of locating 92. Instead of counting from the beginning, she prefers to start from 100 because 100 is nearer to 92 than 0. I argue that the idea of combinations that make ten, i.e. 2 and 8 make ten help Jenna to easily determine that 92 is 8 less than 100. This classroom community has come to a new sophisticated strategy beyond counting by tens in locating numbers on an empty number line, but also employing number relations. The pattern of the string of beads and the activity of locating numbers are proven to support children's thinking, not only in individuals, but also to facilitate the development of classroom mathematical practices.

For more evidence, Fakhri's performance in locating numbers shows that he is supported by the ten-pattern in the string of beads to improve his counting strategy from counting by ones to counting by tens:

The problem: where is the location of 54 on the number line below?

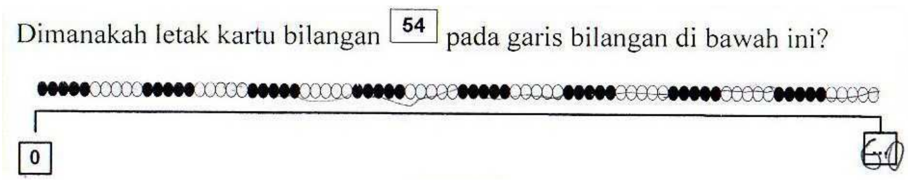


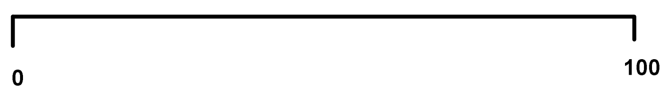
Figure 5.1. Fakhri's performance on his worksheet (14 August 08)

Fakhri wants to know how many beads there are to fill in the blank box at the end of the number line. First, he counts by ones (it is seen from the picture that he tags every bead while he is counting). After the 20th beads, he is encouraged to use the structure of the string of beads to count by tens, because he has many beads left (see the lines that cover every group of ten along the beads from the 21st bead) Then, starting from the 21st bead, he counts by tens: 10, 20, 30, …, 60. Then he fills in the blank box with 60.

The fact above shows that although Fakhri comes to the wrong answer, he is encouraged to count by tens using the structure. Moreover, he realizes that two groups of five beads makes ten.

3) ***Locating numbers on an empty number line without the help of a paper strip***

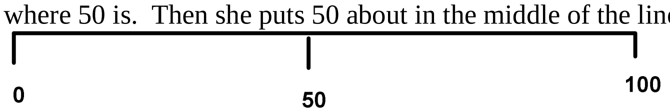
The teacher draws an empty number line and puts number 0 and 100 on both edges.



The teacher : " In the last meeting, we had an almost empty number line with numbers 10, 20, 30, … on it. When you are asked to put 35 on the number line, you simply put 35 in between 30 and 40. Now, I have an empty number line with only the numbers 0 and 100 on it. Who can place the number 50 on the empty number line?"

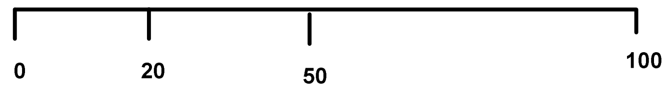
Jenna, Yona, and Dea raise their finger (only a small number of children who dare with this challenge.

Alfi : (She looks at the distance between 0 and 100 and using her span to estimate where 50 is. Then she puts 50 about in the middle of the line.



The teacher : "Is the 50 here?" (pointing to the 50)
The children : "yes!"

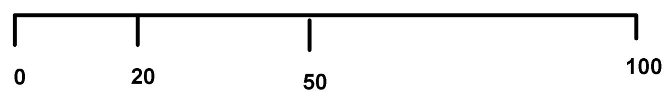
The next child is asked to locate 20 (Shafa):
Shafa : (she is pointing to a point between 0 and 50 but nearer to 0 than 50)



The teacher : "Do you think it is correct or not?"
The children : (most of them say "Wrong!")
The teacher : "To which direction do you have to relocate it?"
The children : "to the left!"

Then Yona is asked to correct the position of twenty:

Yona : (She makes two taps on the number line as if she points to the position of 10 and 20. Then she writes 20 on the second tap, it is more to the left than before)



Then the next child (Annisa) locates 90. She locates 90 near to the 100 and other children count that answer as a correct answer.

Alfi uses her span to estimate the position of 50 and Yona performs two taps from 0 to 20. They might have an image in their mind that there is a string of beads which has a structure of ten in it, or they may imagine the sequence of tens (numbers instead of beads). In spite of this, there is no discussion why Alfi locates 50 on the place where it is. However, they do not need to count by ones again to locate numbers on an empty number line without the help of a paper-string of beads above the line. They also employ their number relations to estimate the position of numbers. It is impossible for them to locate 20 on the right side of 50, because they know that 20 is less than 50, so it must be on the left side of 50 and closer to 0 rather than to 50.

Answers and Recommendations

The description and interpretation above support the conjecture that a locating numbers activity could be used to explore number relations and to enhance children's counting strategies. By understanding number relations, children do not only operate numbers and find the correct answer, but they also realize what they are really doing in performing mental arithmetic strategies on an empty number line, and attempting to find a reasonable answer.

Nevertheless, there is no discussion about children's reasoning in locating number. The teacher only asks for the children's agreement on the correct result without knowing the reason. It would have been better if the teacher asked for the children's reasoning, and their thinking while locating numbers. The children are expected to be able to explain their answers and reasoning, so that their answers make sense for the others.

In a future teaching experiment, one's performance in applying number relation in such activities should be the topic of discussion in a classroom. This will give an idea for other children to also see the relations between numbers, an idea which is crucial in performing mental calculation.

5.3.4. Exploring Addition Strategies on an Empty Number Line

<u>Data Descriptions and Interpretations</u>

1) An empty number line as a model for calculating strategies

After the activity locating numbers on an empty number line, the teacher poses an addition problem to the children, i.e. 33 + 25=

The teacher : "Who could solve this problem using an empty number line?"

(Maudy answers that she could solve the problem, and then the teacher asks Maudy to write her answer on the board)

The teacher : "with jumps of tens!" (gives an instruction to Maudy)

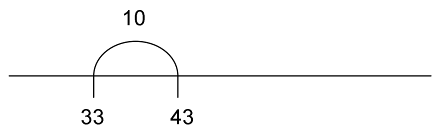
Maudy writes on the whiteboard: 33 + 10 = 43, 43 + 10 = 53

Maudy : " I can not add 10 more"
The teacher : "How many is the remainder when you can't add 10 to it?"
Maudy : "Five!"
The teacher : "Five! Okay, good!"

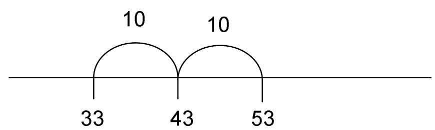
Then Maudy continues writing 53 + 5 = 58 next to the sequence already there.

The teacher : "Do you all understand what Maudy is doing?" (asks the others)
The teacher : " So, if we want to use an empty number line…" (drawing an empty number line)
" Look, Maudy wants to add 10 to 33. How many is it now?"
(while drawing a jump of ten on the empty number line from 33 to 43)

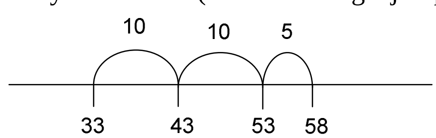
The children : "Forty three!" (only some of them shouting the answer)



The teacher : "Okay, look at this, I just want to replace what Maudy is doing on an empty number line"
" How many more do you have to add?"
The children : "Ten!" (more children are shouting than before)
The teacher : "So, how many is it now?"
The children : "Fifty three" (only a small number of children who shout the answer, and the teacher continues drawing another jump of ten)



The teacher : "Maudy said that there is no more 10 left, so how many is the remainder now?"
The children : "Five" (only a small number of children who answer)
The teacher : "So, how many is it now?" (while drawing a jump of 5)



The children : "Fifty eight"
The teacher : "Can you do this?"
The children : "Yes!"

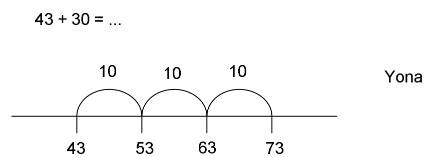
The data description above explains how an empty number line is introduced by the teacher to solve addition problems. Maudy can solve the 33 + 25 problem using the 'jumps of ten' strategy, then the teacher gives a visual representation of Maudy's strategy on an empty number line with the intention that all children can understand what Maudy is doing when adding 33 + 25.

Although the planned instruction in the HLT was not to give direct instruction in applying the 'jumps of ten' strategy, the teacher does give direct instruction for Maudy to use the 'jumps of ten' strategy in solving the problem. This instruction gives an idea for Maudy and the other children also that the teacher only wants them to use the 'jumps of ten' strategy.

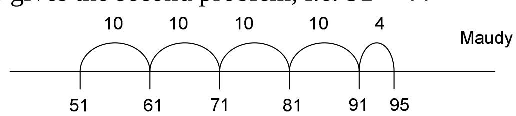
However, the 'jumps of ten' strategy which requires decomposing the add-end number can improve children's number relation, i.e. 25 can be decomposed into 10, 10, and 5. The empty number line as a model for calculating and the role of the teacher in drawing the jumps one by one also help children to realize that every jump of ten means adding a ten to the previous number. Moreover, Maudy has given a contribution to the classroom in giving another strategy in solving addition problems besides the formal strategy that the children always count on. The classroom community, then, reorganizes their thinking that in solving addition problems they can apply another acceptable mathematical solution, i.e. the 'jumps of tens' strategy.

2) Exploring addition strategies on an empty number line

The teacher opens a discussion in the classroom by giving an addition problem that has been solved in the last meeting. First, the teacher asks the children to solve 47 + 30 on the whiteboard, and then Yona moves to the front and writes her strategy in solving the problem on an empty number line that has been drawn by the teacher.

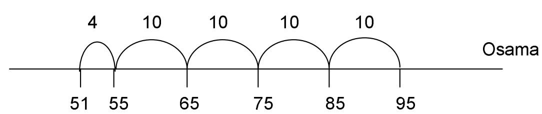


Then the teacher gives the second problem, i.e. 51 + 44 =

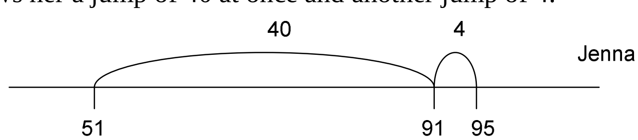


The teacher : "Why is there no a small jump in the first problem?" (a small jump means a jump of a number which is smaller than ten)
Jenna : "Because there is no 'ones' in the first problem"
The teacher : "Good! Now, is there another way to solve this problem?"

Osama is asked to write his answer on the whiteboard:



The teacher asks for another different strategy which has a big jump, and then Jenna shows her a jump of 40 at once and another jump of 4.



The teacher : "Which one is easier for you? Now look at these answers, do they all have the same answer?"
The children : "yes"
The teacher : "Do they use the same strategy?"
The children : "No, different"
The teacher : "What strategy does Maudy use?"
The children : "jumps of tens"
The teacher : "How many jumps does she perform?"
The children : "four"
The teacher : "Why four?"

The children	: "Because there are four tens"
The teacher	: "Good. Do you understand? What is the difference between Maudy and Osama's strategy?"
Alfi	: "There is a difference in the 'ones'"
The teacher	: "What happens to the 'ones' in Osama's strategy?"
Alfi	: "The 'ones' are in the front"
The teacher	: "Osama said that it is easier to add 4 to 51, so he starts from 55 and then performs jumps of tens. Now, Jenna has a different strategy, what does Jenna do?"
The children	: "She performs a jump of 40 at once"
The teacher	: "Is it allowed to do a jump of 40 at once?"
The children	: "yes"
The teacher	: "Now, if I label this strategy (Maudy's strategy) as 'a', then this is (Osama's strategy) 'b', and this (Jenna's strategy) is as 'c'. Which one is easier for you?"
The children	: "The 'c'!"

In this activity, the children try to explore different strategies that they could perform on an empty number line to solve an addition problem. The teacher asks the children for another different strategy, which invites the children to think of another efficient strategy. In this respect, the teacher plays an important role in enhancing children's thinking. The teacher not only asks for a different strategy, but also emphasizes the difference among the strategies.

The classroom community comes to a taken-as-shared way of performing the more efficient strategy in solving addition problems on an empty number line. They agree that Jenna's strategy is more efficient than the other strategies. Jenna has made a significant contribution to the classroom community for the idea that the bigger the jump is the more efficient the strategy is. In other words, the lower the number of jumps is, the more efficient the strategy is.

Some children's work on paper that was captured by the video:

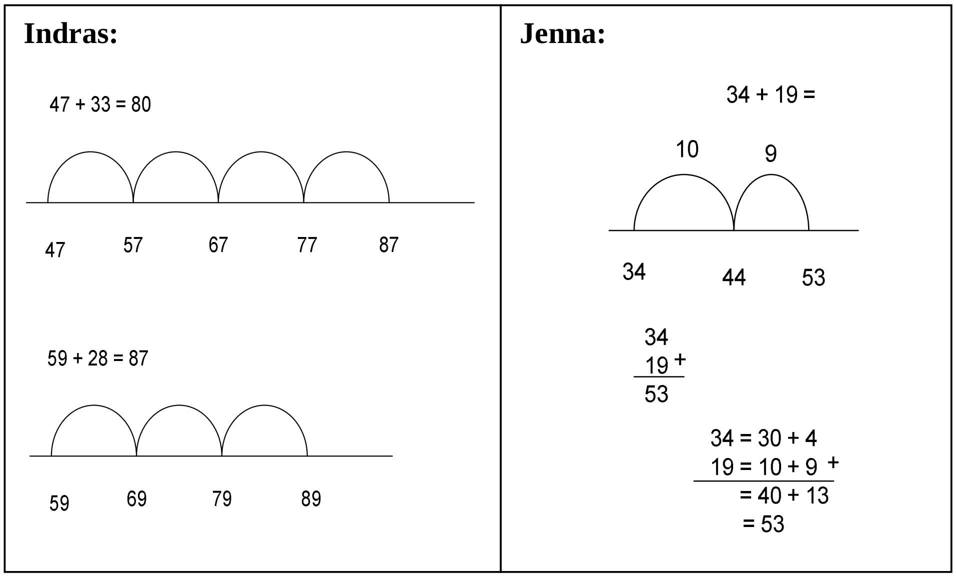


Figure 5.2. Indras and Jenna's strategy in solving addition problems (15 August 08)

The figure above explains us that the 'jumps of ten' strategy is not meaningful for some children including Indras. He knows that every time he performs a jump means adding a ten to the number consecutively. He can solve the problems '47+33=80' and '59+28=87' correctly although we do not know for sure the strategy he uses. But the instruction of using an empty number line to solve the problems put Indras in trouble when he did not understand how to work with it. The empty number line is not meaningful for Indras because it does not represent his mental strategies in solving problems. This might happen because there is no discussion on the problem strings given. The teacher is supposed to pose the problem strings one at a time, which requires children to count mentally and then discuss their strategies. In fact, the teacher just writes down all the problems on the board and asks the children to solve them in their book.

On the other hand, Jenna gives three different strategies in solving addition problems. All three strategies give a correct answer because Jenna is one of the high level children in the classroom. When I interviewed Jenna, she showed a good understanding of what she is doing with all the strategies she performs. It seems that Jenna has no problem in dealing with addition problems using different strategies.

Answers and Recommendations

An answer to the third research question regarding the role of the teacher could be given based on the data above. As in traditional mathematics teaching, where the children are expected to give answers as the teacher wishes, the teacher still plays a role. Instead of giving the freedom to solve problems in the children's own way, the teacher gives a direct instruction to use the 'jumps of ten' strategy that can limit children's thinking in doing mental arithmetic. Moreover, the direct instruction of 'using an empty number line to solve problems' puts the empty number line into less sophisticated and meaningless strategy use than expected. Children like Indras perform the addition problem on an empty number line without understanding.

The problem string which was intended to practice children's mental calculation is not helpful, because the teacher asks the children to solve the problems in their book. It will be helpful for children to practice their mental calculation by answering the problem string one by one and discussing their different strategies in their head. The idea is to employ number relations in performing mental arithmetic instead of drawing an empty number line.

This could also answer research question (2.4) on the development of children's mental arithmetic supported by the empty number line. The empty number line cannot support the development of children's mental arithmetic when the idea of number relations in calculating mentally is neglected in the classroom discussion.

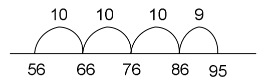
5.3.5. Developing Addition Strategies on an Empty Number Line

Data Descriptions and Interpretations

1) Comparing more efficient strategies in solving certain addition problems

The teacher starts a discussion in the classroom by posing a problem on the white board, i.e. 56 + 39 = …
Fakhri is asked to show his strategy on an empty number line in solving the problem. The teacher has drawn a line under the problem, and then Fakhri starts solving the problem by writing 56 on the empty number line. Then he performs three jumps of ten until he gets to 86. Fakhri realizes that the last jump must be a jump of 9, and he counts one by one using his finger to find out the result of 86 + 9 which is 95. After that, he writes the explanation of his answer under his number line. The picture below illustrates Fakhri's answer on the whiteboard:

56 + 39 = ...



56 + 10 = 66
66 + 10 = 76
79 + 10 = 86
86 + 9 = 95

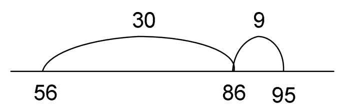
The first part of the classroom mathematical activity above shows us that Fakhri can show how to perform his mental calculation on the number line. He can solve the problem by performing the 'jumps of ten' strategy. He employs his understanding in counting by tens (56+10=66; 66+10=76; etc) and decomposing a number to solve the problem. Fakhri's decision in performing a jump of 9 at the end of his calculation on an empty number line shows us that he understands that the decomposition of 39 is 10, 10, 10, and 9. Although at the beginning he miswrote the 76, he corrects himself later after the classroom community discusses his strategy in solving the problem. Furthermore, we observed that Fakhri applies counting by ones while adding 9 to 86. That tells us about Fakhri's understanding of number relations. He does not use the number relation that 9 is one less than 10, so he could just have add 10 to 86 and subtracted one.

The teacher	: "Ok, good! Now, let's see Fakhri's answer. Is it correct?"
The children	: "yes!"
The teacher	: "Is it correct?"
The children	: "yes!"
The teacher	: "What kind of jumps did Fakhri do?"
The children	: "jumps of ten"
The teacher	: "Why in the last jump that Fakhri only wrote 9?"
The children	: "Because the problem asks for 39"
The teacher	: "Do you understand?"
The children	: "yes!"
A child	: "Fakhri made a mistake in writing 79 + 10 = 86"
The teacher	: (points to the 79) "Okay, what it is supposed to be?"
Fakhri	: "Oya, it should be 76!"
The teacher	: "Good, you can correct it by yourself"

Then Fakhri moves to the whiteboard and changes 79 to 76 in his answer.

The teacher plays her role in facilitating the children's discussion in the classroom. She repeatedly asks the children "Is it correct?" to encourage children's participation in the classroom as validators. The classroom community judges that Fakhri's solution is an acceptable answer for them, until a child says that Fakhri made a mistake in the '79' that is supposed to be '76'. The classroom discussion brings Fakhri to correct his own mistake. The classroom community not only acts as a validator for an acceptable mathematical solution but also contributes to the development of the children's thinking at the same time. Moreover, the teacher's instruction for Fakhri to correct his own mistake develops his responsibility

The teacher : "Ok, who has the same answer as Fakhri?"
(Some children raise their hands)
The teacher : "Who has different answers than Fakhri?"
(Some children raise their hands, and then Kiki is chosen to give her answer)
After the teacher draws a line under the first answer, Kiki starts from 56 and performs a jump of 30 at a time to get to 86. Then she performs another jump of 9 from 86 to 95 (see the picture below).



56 + 30 = 86
86 + 9 = 95

The teacher : "Kiki performed a jump of 30! Great! This is a different strategy. Is there any other strategy?"

The teacher's question for another strategy demands the children to come up with different strategies than before. The question is in fact intended to foster the emergence of the mathematical practice in comparing more effective strategies for certain addition problems. The problem 56 + 39 was chosen intentionally for the emergence of the 'jumps *via* tens' strategy using a compensation technique, in which 40 is added to 56 and then the compensation is subtracting the result by one. Kiki comes up with the more effective strategy than Fakhri, because she can do a jump of 30 at once while Fakhri needs three jumps of ten. The teacher counts Kiki's answer as a different solution than before. However, the difference between the two solutions presented is not made clear by the classroom community. The teacher immediately asks for another different strategy and gets a reply from Shafa.

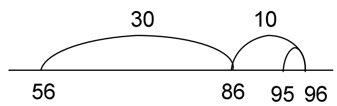
Shafa raises her hand and she gives her answer on the whiteboard. She performs a jump of 30 at once from 56 to 86. Then she performs a jump of 10. Before Shafa finishes her answer, there is a child who says that it is impossible to perform a jump of ten. But the teacher says to the children to allow Shafa to finish her answer. Then Shafa continues writing 96 at the end of the second jump and she performs another little jump backward.

The other children are surprised with Shafa's backward jump: "Hah!", "I don't use a jump backward", and so on. But Shafa gets confused and the teacher tries to help her by posing questions:

The teacher : "How many have you added to the number?"

(Shafa seems confused)
The teacher : "Here, look at your jumps, you have a jump of 30 and a jump of 10, how many do you need to add in the problem?"

The teacher poses the question not only to Shafa, but also to the other children in the classroom. There is a child who says that Shafa needs to perform a jump of one backward, and the teacher thanks the child for the answer. Then Shafa continues her strategy on the empty number line by performing a jump of one backward from 96 to 95.



56 + 30 = 86
86 + 10 = 96
96 – 1 = 95

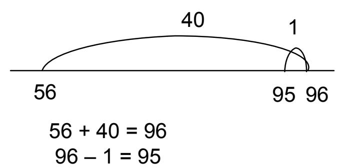
The sequel of the discussion above tells us that the classroom community really supports Shafa's thinking process in solving the problem using number relations. On the other hand, Shafa's strategy contributes to the development of socio-mathematical norms in the classroom by providing a different strategy that draws children's attention in validating whether Shafa's strategy is an acceptable mathematical solution or not.

Shafa has an idea that it is possible to perform a jump backward when the add-end number is less than the total number on the jumps. She might get confused after a child says that it is impossible to perform a jump backward. That statement by another child influences Shafa's mathematical reasoning at the beginning, although at the end she shows her good understanding of number relations by performing a jump of one backward. Shafa reorganizes her thinking during the discussion. Shafa's thinking is supported by another child's contribution when the child mentions that Shafa needs to perform a jump of one backward. The teacher's appreciation for the child's contribution to the development of the mathematical practice strengthens the classroom social norms in the classroom as well as the children's beliefs about explaining their thinking. The mathematical explanation under the empty number line – 56+30=86; 86+10=96; 96-1=95 –

helps children to understand what counted as a different strategy and the involved number relations.

After several strategies have been performed, the teacher poses a question about the easier strategy and most of the children shout that the strategy with a jump of 30 is the easier one. Then the teacher tries to encourage children to find more efficient strategies by posing a question:

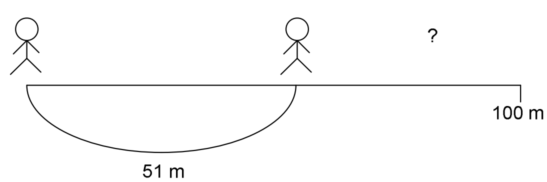
The teacher : "Can we perform a jump of 40?"
Yona : "Yes, we can. Then we have to subtract one"
The picture below illustrates Yona's solution on the whiteboard.



Facilitating a discussion by comparing strategies, and posing an important question to stimulate children's thinking to find the more effective strategy are parts of the role of the teacher that is explained by the data above. Yona as a smart student in the classroom can show her good understanding of number relations in solving the problem. She understands that 39 is one less than 40, so she has to perform a jump of one backward after performing a jump of 40 at a time. Yona's contribution to the classroom mathematical practices reorganizes children's mathematical reasoning of what counted as a more effective strategy.

2) Missing add-end problem

The teacher poses a context problem about a sprinter who has been running for 51 m. The sprinter has to run for 100 m, how many meters more does he have to run to reach the finish? (The teacher poses the problem by giving a visual representation of the problem, see the picture below)



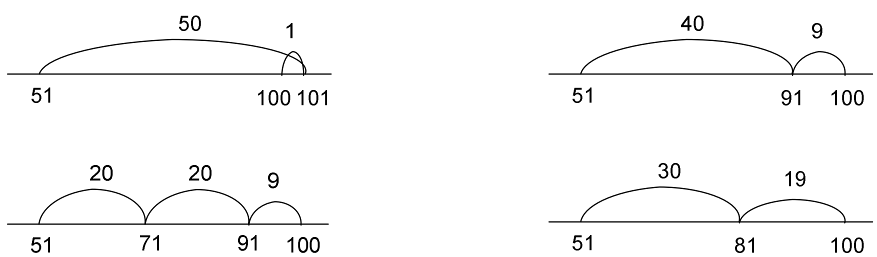
The children answer spontaneously: 50! (There is an answer from one of the children that mentions 49).
Then the teacher repeats the problem again, but the children are still quiet without an answer. This condition brings the teacher to pose another similar problem which is easier for the children.

The teacher: "I have 20 longan (kelengkeng) fruits, and I have eaten 11 of them. How many longans more should I eat to finish them up?"

The teacher asks the children to use an empty number line to find the answer, and then the teacher draws an empty number line to help the children solve the problem. A child answers that they have to do a jump of nine. After that, the teacher comes back to the first problem about the sprinter.

The teacher: "How do you count to solve the problem? Now you use an empty number line to solve the problem on a piece of paper! Make a drawing like on the whiteboard!"

While the children solve the problem on their paper, some of them give their answers on the whiteboard. Below are their strategies on an empty number line:



The children can perform the 'jumps of tens' strategy to solve the problem, but they don't write how they come to the answer, which is 49. Therefore, the teacher confirms the children's answers by asking a question: "What is your answer to this problem? How did you get 49?"
The children get the 49 by adding the numbers on top of the jumps that they performed.

The missing add-end context problem was intentionally proposed to bridge the relation between addition and subtraction in which children realize a subtraction problem as an adding on problem. The context problem is chosen so that children will use their informal strategy in solving the problem instead of the algorithm. In fact, the teacher gives a direct instruction for the children to use an empty number line in solving the problem.

Below are some examples of children's written assessment in handling the missing add-end problem:

The problem: *In the competition for Indonesian Independence Day on the 17th of August, Annisa has to collect 50 hidden Indonesian flags. Annisa already has 19 flags with her. How many more flags does she have to find?*

The context problem below does not give a direct instruction to use an empty number line. The children are free to choose strategies that fit them.

Alfi's strategy represents four other children in the classroom, who are solving this problem correctly using an algorithm strategy. The picture below suggests to us that the empty number line is not meaningful for Alfi. Alfi knows the answer is 31, but she has no ideas for using the 'empty number line strategy' (this is what children often mention). The empty number line does not represent Alfi's thinking in solving the problem.

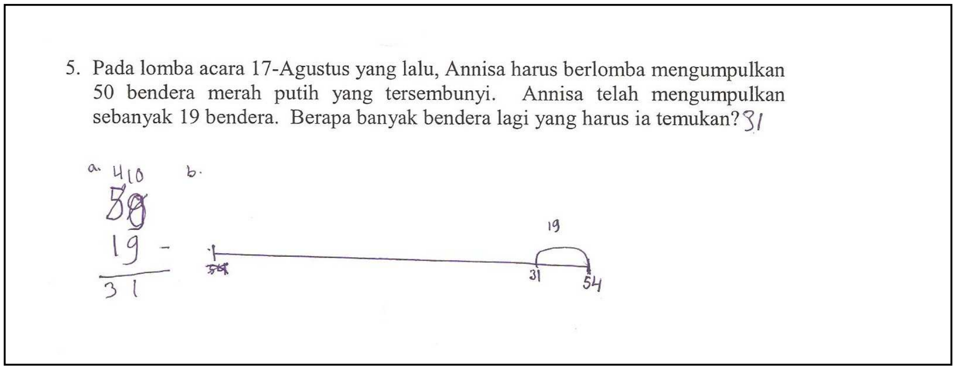


Figure 5.3. Alfi's strategy in the contextual missing add-end problem

Below is Maudy's solution for the problem. Three other children have interpreted the problem in the same way as Maudy. They perform correct jumps in performing the 'adding on' strategy on the empty number line. But they do not understand the idea that the total number on every jump corresponds to the result. They just perform the jumps without a great deal of thought. Therefore, they cannot reach the answer, which is 31; in Maudy's answer, 31 is obtained from 20 + 10 + 1 (the total number on every jump).

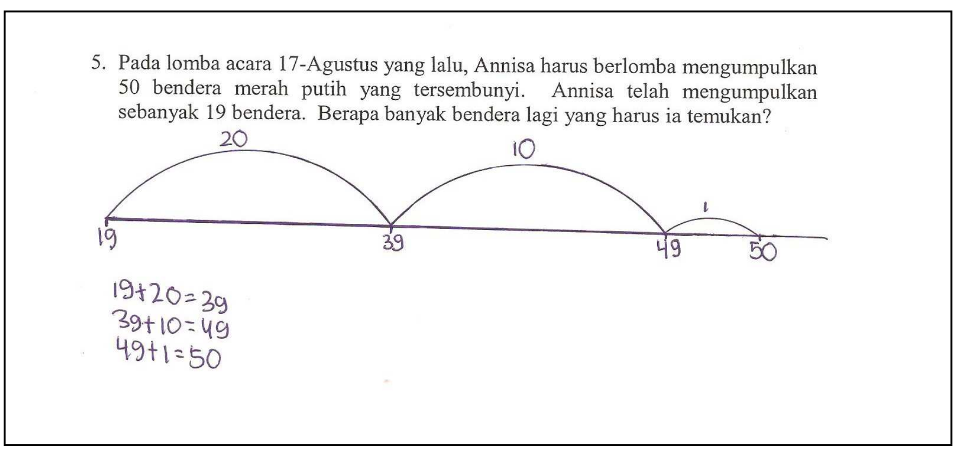


Figure 5.4. Maudy's strategy in the contextual missing add-end problem

However, three children, i.e. Rezon, Rizqy, Yona, and Dhea show their understanding in solving this problem on an empty number line. The empty number line represents their thinking in finding a solution for the problem. Yona's thinking process is discussed more elaborately in the next sections of this chapter.

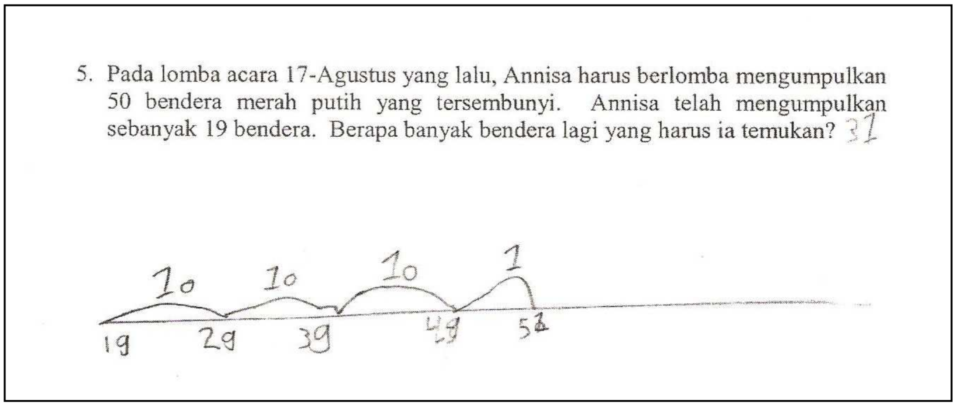

5. Pada lomba acara 17-Agustus yang lalu, Annisa harus berlomba mengumpulkan 50 bendera merah putih yang tersembunyi. Annisa telah mengumpulkan sebanyak 19 bendera. Berapa banyak bendera lagi yang harus ia temukan? 31

Figure 5.5. Rezon's strategy in the contextual missing add-end problem

Rezon's performance in the activity of locating numbers shows that he has a good understanding of number relations. That makes me look at his answer on the final assessment on context problem. While other children have difficulties in solving this problem, Rezon does not. We could see from his answer above that he uses an empty number line to find the solution by the adding on strategy. He knows that the accumulation numbers on entire jumps that he performs – 10, 10, 10, and 1 – is the number of flags that Annisa has to find in the problem. He is not only able to see the number relations between adjacent numbers, 19 + 10 = 29; 29 + 10 = 39; …; 49 + 1 = 50, but also to understand that his adding on strategy should end at 50 and the total number of jumps makes 31.

Out of 35 children in the classroom, there are more than 20 children who cannot give a correct answer for this problem. Some of them make the common mistake in the algorithm strategy for subtraction (50 – 19 = 49), some of them add the numbers 19 and 50 instead of finding the difference (19 + 50 = 69), and some of them try to give an answer on an empty number line without understanding. This means that the empty number line is not yet a model for the children in representing their flexible mental strategies.

Answers and Recommendations

The analysis of the emergence of the classroom mathematical practice of developing more efficient strategies in solving addition problems encompasses not only children's development as an individual, but also their contribution in the development of the mathematical practices for the whole class. Some research questions regarding the empty number line, the development of mental arithmetic strategies, the role of the teacher and the classroom culture can be answered based on the data description above.

The episodes of the classroom mathematical activity show us that the children involved in the teaching experiment can develop their mental calculation in solving addition problems using an empty number line. This fact answers research question (2.4). The empty number line provides a visual representation of children's calculating strategy that can be perceived and discussed by the classroom community. The empty number line facilitates children's discussions in the classroom about one's mistakes and strategy. The empty number line has also proved it can develop more sophisticated strategies in solving certain problems. For example, in solving 56 + 39, Yona can come up with a jump of 40 strategy followed by a jump of one backward, which is the most proper strategy for this kind of problem. It takes much time when children try to solve this problem by operating the digits, because the problem requires a conversion.

Furthermore, the empty number line provides the improvement of more sophisticated strategies for mental calculation in solving missing add-end problems using the 'adding on' strategy. Instead of performing an algorithm to solve 100 – 51 or 50 – 19, the adding on strategy on an empty number line is proved powerful in avoiding the common mistakes in the algorithm and gives another countable mathematical solution (see the analysis of Yona for her strategy in solving the contextual missing add-end problem).

From the 'jumps of tens' and 'jumps via tens' strategy, children develop their flexibility in performing the calculation. The classroom community discusses the more efficient strategy which brings several different strategies in solving certain problems. The role of the teacher in posing context problems, posing questions to manage mistakes, and facilitating children's discussions, is another factor that develops children's flexibility in mental calculation. This explanation can answer research question (1.2) regarding children's flexibility in mental arithmetic strategies. However, there is still a tendency that children always follow what the teacher instructs them to do. The children want to make the teacher satisfied with their answers. Moreover, the teacher sometimes misses the children's reasoning or does not find out exactly what the children's reasoning is.

There are still some children who do not develop their understanding in constructing mental arithmetic strategies on an empty number line. They find difficulties in dealing with the 'empty number line' strategy, because the empty number line does not make sense for them when they are able to solve addition problems using their number sense or the algorithm strategy. The children's performance in the classroom and in their written assessment shows that they perceive the empty number line as a newly taught procedural calculation strategy in solving addition problems, in which the procedure is performing jumps of tens starting from the first number. The empty number line is not yet a model for the children's mathematical reasoning that they can use flexibly. Interviews during a classroom activity prove that the children do the 'empty number line' strategy because the teacher wants them to do so. This concurrently answers the question regarding the role of the teacher and question (1.3).

The teacher's belief about mathematics education in the classroom has an effect on the development of children's thinking in constructing mental arithmetic strategies. Giving direct instructions on using an empty number line somehow limits children's freedom in doing mathematics in their own way which contradicts the RME principle. However, the teacher helps children in their development by facilitating discussion; building a classroom culture in which children can participate in the classroom as validators; and encouraging children to contribute to the classroom discussion for the sake of the development of their learning process.

The results and analyses of the teaching experiment above lead to some points for recommendations. The next teaching experiment should focus on the number relations to perform mental calculation instead of the empty number line as a new strategy to calculate addition problems. Children's performances in the contextual missing add-end problem show that they focus too much on performing jumps of tens without an understanding of the whole process, i.e. the decomposition of the second number. The teacher plays a significant role in guiding children to develop their mathematical understanding. The children might need more time and more different contextual problems besides influencing the role of the teacher to be more interrogative and open to children's thinking. Therefore, a negotiation process with the teacher should be conducted regularly to improve the teacher's beliefs about mathematics teaching and learning.

5.3.6. Yona's Learning Process

Data Descriptions and Interpretations

Yona is one of the best students in class 2A. She always pay attention to every lesson and is very active in the classroom. The written final assessment below is used to show the development of Yona's thinking. Yona's performance in the classroom during the teaching experiment has been embedded in the previous sections.

1) Written Final Assessment – no.5

Problem: *In the competition for Indonesian Independence Day on the 17th of August, Annisa has to collect 50 hidden Indonesian flags. Annisa already has 19 flags with her. How many more flags does she have to find?*

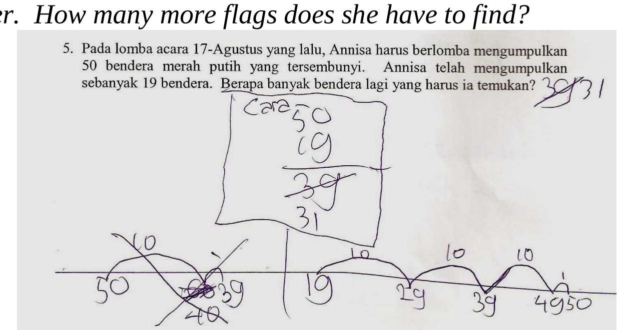


Figure 5.6. Yona's written assessment for no.5 (21 August 08)

First, she realizes that it is a subtraction problem. Therefore she performs an algorithm strategy to find the solution of the problem. As is shown in figure 5.6., Yona makes a common mistake in performing the algorithm procedure. She subtracts 9 from 0 (the second digits) as she subtracts 0 from 9, then she subtracts 1 from 5 as if she borrows a ten for the second digit. Second, she writes down 39 as the answer right after the question mark of the problem.

Third, she wants to justify her answer by performing calculation on the empty number line (see the left side). She draws an empty number line and writes 50 on the left side and draws a jump (a curve line) of ten to the right. At first, she though that she is adding 10 to 50, but she realizes that she is doing subtraction. Therefore, she crosses the 60 and 40 replaces the 60. She draws another smaller jump to go to the 39 (because she knows from the algorithm that the answer is 39). The mark on top of the smaller jump indicates that she was going to write the number 1, but she suddenly stops and realizes that it is not supposed to be 1. When she adds the 10 and 1 together, she won't get 19 as the number that is subtracted from 50 in the problem.

Fourth, she realizes her mistake and crosses her first number line. Fifth, she gets 31 as the correct answer either by performing the algorithm once again (correcting 39 to be 31) or by calculating on the empty number line on the right side. If we observe thoroughly how Yona works on an empty number line (see figures 5.8, 5.9, 5.11), she is able to make a jump of 20 or 40 at once on an empty number line, while in fact she makes very careful jumps of tens in this case. She could have performed a jump of thirty at once if she already knew the answer from the algorithm (if she corrects the algorithm first). Therefore, she performs the calculation on the empty number line first, before correcting the algorithm. She ends by correcting her mistake by crossing out 39 (the first answer) and writes 31 next to 39.

2) Written Final Assessment – no.6e

The instruction of this problem is: *How do you solve this problem on an empty number line?*

6.e. 34 + … = 61

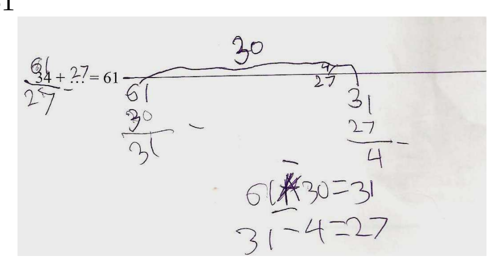


Figure 5.7. Yona's written assessment for no.6e (21 August 08)

In the missing-addend problem above, Yona realizes it as a subtraction problem. Therefore she calculates using the algorithm to find the answer. As is shown in figure 5.7., Yona seems to perform the algorithm first and then calculates on an empty number line, followed by adjusting the jumps to get the same result.

Since Yona recognizes the problem as a subtraction, she proceeds with the subtraction as taking away in which she starts from 61 and makes a jump of 30 to the right (taking away 30 from 61), and gets 31.

The problem occurs when Yona makes a second jump to go to 27 from 31. She thinks that she could perform another jump of 4 in the opposite direction of the first jump as she usually does in the addition problem. She should have made the second jump to the right, not to the left to take away another 4 from 31. Although she gets the correct result, she does not understand how to deal with the empty number line for a subtraction problem as a taking away problem. This could have happened because the discussion in the classroom only went as far as addition on an empty number line and the missing-addend problem, using the adding on strategy on an empty number line, not including subtraction as taking away.

As is shown in figure 5.7., Yona represents her thinking in subtracting 30 from 61 and 27 from 31 as also using the algorithm procedure. It seems that the empty number line is used only to accomplish the instruction of this problem. If problem no.5 (figure 5.6.) and problem no.6-e (figure 5.7) are compared, both are similar problems. While problem no.5 is a context problem, 6-e is a bare number problem. It could be that the problem with less context is less meaningful for children.

3) Written Final Assessment – no.6a

The instruction of this problem is: *How do you solve this problem on an empty number line?*

41 + 19 =

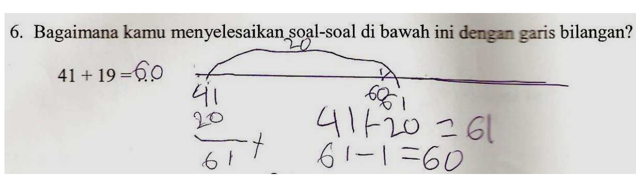


Figure 5.8. Yona's written assessment for no.6a (21 August 08)

In the picture, Yona shows her flexibility in doing addition on an empty number line. She starts writing 41 as the fixed number and doing a jump of 20 at once. Since she knows that 19 is 1 less than 20, she performs another opposite jump to go one step back from 61 to 60.

Under the number 41, Yona shows her strategy of how to get 61 from 41 plus 20 by algorithm. It seems useless, because children at Yona's level do not need to perform an algorithm to find the answer of 41 + 20. From an interview and a classroom observation, Yona knows the place value pattern very well, therefore answering 41 plus 20 is an automation for her. The next two mathematical sentences (41 + 20 = 61 and 61 -1 = 60) explain how she performs jumps on the empty number line.

4) Written Final Assessment – no.6b

The instruction of this problem is: *How do you solve this problem on an empty number line?*
24 + 65 =

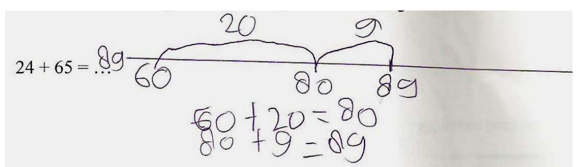


Figure 5.9. Yona's written assessment for no.6b (21 August 08)

Yona probably uses splitting in her mental calculation, i.e. separating the tens and ones, then operating the tens and ones separately, finally adding the tens and the ones to get the result. She is very flexible in this case, because she prefers the splitting strategy for the addition problem which doesn't require conversion of tens. In this case, Yona shows how she represents her thinking on the empty number line by adding 20 to 60. She prefers to put 60, then 20 as the first start, because adding 20 to 60 is much easier than adding 60 to 20. Then she adds the 9 (she put 4 and 5 together) to go to 89 from 80. Actually, the splitting strategy has never been discussed in the classroom; therefore it is a surprise that Yona can come up with the splitting strategy on an empty number line.

Another child, Rizqy also shows his flexibility in dealing with this problem (see figure 5.10 below). He add the ones first to the first number and then add the tens – two jumps of 30 – to get the result. In this case he performs a *mixed method* in which he separates 65 into 60 and 5, then proceeds them consecutively from adding the '5' and then adding the '60' with two jumps of 30.

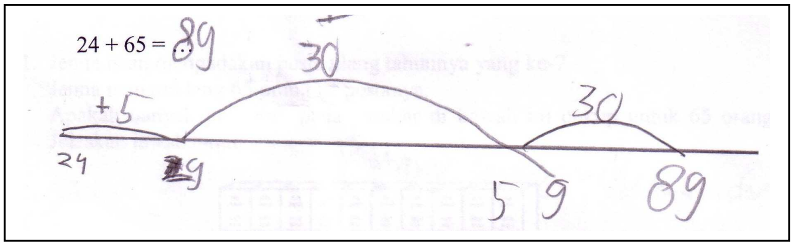


Figure 5.10. Rizqy's written final assessment no 6b (21 August 08)

5) Written Final Assessment – no.6d

The instruction of this problem is: *How do you solve this problem on an empty number line?*
39 + 48 =

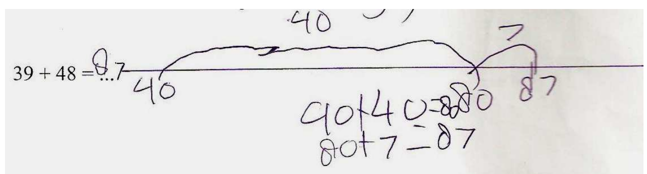


Figure 5.11. Yona's written assessment for no.6d (21 August 08)

This problem shows another flexible solution from Yona in doing addition on an empty number line. She solves 39 + 48 by first doing the mental calculation that 39 + 1 is 40, so she just has to start from 40 and continue with a big jump of 40 (from the 48) to go to 80. Since one is taken from 48, she only has to add 7 more to 80 to get the result of the problem.

<u>Answers and Recommendations</u>

The explanation of Yona's performance in her written final assessment and her interview provides answers to the research question (1.2), (1.3), (2.3), (2.4), and (3). The data of 5.3.6 (3, 4, 5) on Yona's written assessment show us her flexibility in performing mental arithmetic strategies on an empty number line (an answer to question 1.2). Although in some parts of this calculation she still uses algorithm either to find an answer or to check an answer, she shows an understanding of doing mental calculation on an empty number line. She makes use of the empty number line as a model for her mathematical reasoning (an answer to the question 2.4). For example, she prefers the splitting strategy in solving 24 + 65 and starts from 60 instead of doing common jumps of ten from the first number as the fixed number on an empty number line. She is able to represent her thinking in the splitting strategy on an empty number line, although it has never been taught in the classroom. This example answers the research question that children could be very flexible in performing mental calculation on an empty number line, using not only the counting on strategy, but also the splitting strategy.

However, the empty number line could also seem very procedural for children (an answer to the question 1.3). Yona tries to solve the bare missing add-end problem (figure 5.7) using an empty number line and an algorithm at the same time. She tries the usual procedure, but she fails to manage it correctly. This may be because she needs more practice in dealing with this kind of problem.

In addition, for the contextual missing add-end problem (figure 5.6) Yona makes use of the empty number line to manage her mistake in the algorithm procedure that she performed. It proves that the empty number line acts as a representation of children's thinking that could help them to trace and realize a mistake in a mental calculation strategy (Gravemeijer, 1994a). The empty number line could also serve as a model for the children's flexibility and their number relations in solving addition problems. These answer questions (2.4) and (2.3).

5.3.7. Annisa's Learning Process

<u>Data Descriptions and Interpretations</u>

1) An interview during the teaching experiment

Annisa has solved an addition problem using the algorithm strategy and now she is trying to solve the same addition problem on an empty number line.

This is written in Annisa's paper:

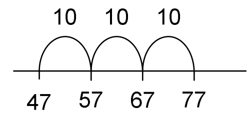
$$\begin{array}{r} 1 \\ 59 \\ 28 \\ \hline 87 \end{array} +$$

Me : "How do you solve the problem?"
Annisa : "With the column strategy"
Me : "Explain to me how you did that!"
Annisa : " I add 9 and 8 then I get 7 here, I put 1 on top of 5, then I count 1 plus 5 plus 2 is 8. So it is 87!"

Me : "Why did you put 1 on top of 5?"
Annisa: "I don't know"
Me : "Ok, Now you explain me how you did the problem no.1!"

47 + 30 = 77

$$\begin{array}{r} 47 \\ 30 + \\ \hline 77 \end{array}$$



Me : "Can you read your strategy on the number line?"
Annisa : " 47 plus 10 is 57. 57 plus 10 is 67. 67 plus 10 is 77.
Me : "So, how many is 47 plus 30?"
Annisa : "Seventy seven"
Me : "Can you do it the same way for 59 plus 28?"
Annisa: "What should I put here?" (pointing to the beginning of an empty number line)
Me : "What do you want to add?"
Annisa : "Eighty…seven…plus…twenty eight…?"
Me : "Can you find the problem 87 + 28 on the whiteboard?" (All addition problems are written on the whiteboard)
Annisa : "Yes…mmm…eighty seven…plus…fifty nine…
Me : "What did you put here?" (I am pointing to her number line on no.1)

Annisa does not really understand what she is doing in solving addition problems using the algorithm strategy. Moreover, her question *"What should I put here?"* shows us that she has no clue in dealing with the empty number line to solve that problem. She does not have a mental calculation in her head that could be represented on an empty number line, therefore she asked *"What should I put here?"* This is an indication that Annisa might think that she has to master a new procedure using an empty number line as the teacher wishes.

Reflecting on the first problem, in which she can perform the addition of 47 + 30 on an empty number line correctly, does not help her realize what she is doing on the number line. This happens because the solution of 47 + 30 on an empty number line has been written on the whiteboard before; therefore she might just copy the answer. Moreover, she always performs the algorithm strategy in solving addition problems, so it seems that she relies on this strategy very much.

2) Written Final Assessment

Problem Instruction: Solve these problems using an empty number line!

Annisa started from 41 and made a jump of 10 to get to 51. Then, she performed a jump of 8 to get to 60 as the final result. She wrote 60 as the final result next to the problem. A sentence written on this problem is "because the number-jump is appropriate for the number below".

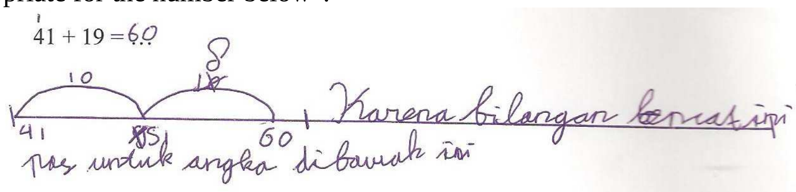


Figure 5.12. Annisa's written assessment for no.6a (21 August 08)

Although Annisa gets a correct result for this addition problem, she makes a mistake when performing the second jump, i.e. the jump of eight to get to 60 from 51. This indicates that Annisa doesn't have a good number sense in doing calculation, because she doesn't recognize that 51 and 8 makes 59 instead of 60, and the decomposition of 19 should be 10 and 9 instead of 10 and 8.

On top of that, we can see a number mark '1' on top of 4 in the problem. Based on an interview with Annisa during the teaching experiment, I assume that she performs the algorithm strategy first to get the result '60' before she draws an empty number line. Because the number '1' suggests us that she adds 9 and 1 (the second digit of both number, i.e. 41 and 19) to get to ten, then she put the 'one' ten on top of the tens column (she almost does the algorithm mentally). After that she adds 1 and 4 from 41, and 1 from 19 to get to 6. Finally, she gets 60 as the answer! And to carry out the strategy which is instructed, she performs a calculation on an empty number line, adjusting with the first result that she gets from the algorithm strategy. This interpretation is also supported by her reason: *"because the number-jump is appropriate for the number below"*, which means that the jumps of ten and another jump of eight are appropriate for the result. In other words, she knows the answer '60' first, then she thinks of the appropriate numbers for the jumps.

This algorithm gives a correct answer. However, the mistake (adding 8 to 51 to get to 60) is an indication that obtaining the correct result using the algorithm strategy doesn't mean that the number relation is involved in the thinking process. Therefore, Annisa has not understood the number relations that could help her performing mental arithmetic strategies to solve problems.

She started from 24 and continuously performed five jumps of ten to get to 74. Finally, she adds 15 more to 74 to get to 89.

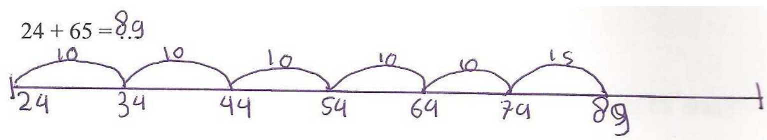


Figure 5.13. Annisa's written assessment for no.6b (21 August 08)

In this problem, she might still use the algorithm strategy to get the answer, i.e. 89. Since the addition of 4 and 5 (the ones from 24 and 65) is not bigger than ten, she does not need to put a number mark "1" to represent a ten on top of the tens column. The last jump - a jump of 15 - is the biggest jump that she does for all the problems in this section. She might have known the result first by doing the algorithm and then adjusted the jumps to come to the correct answer.

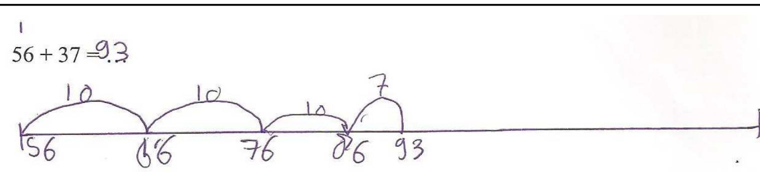


Figure 5.14. Annisa's written assessment for no.6c (21 August 08)

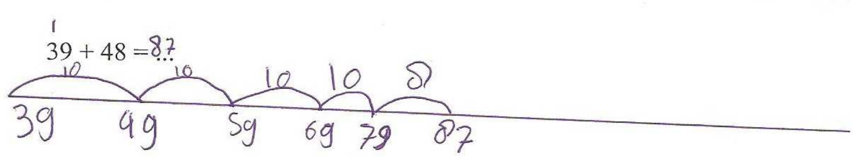
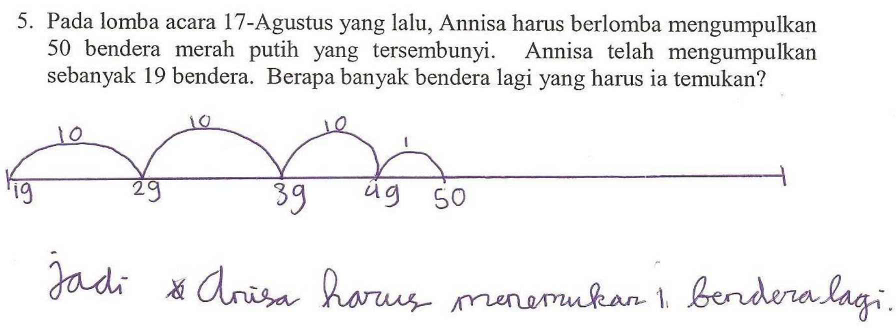


Figure 5.15. Annisa's written assessment for no.6d (21 August 08)

Annisa tends to perform a similar strategy on an empty number line over numerous problems in the written assessment. It might happen because she feels safe to continually perform this similar strategy which has been proven correct, or she thinks the empty number line as a fixed procedure, i.e. doing jumps of tens from the first number, in solving addition problems. I would contend that Annisa performs the algorithm strategy first to find the answer and then adjusts the last jump on the empty number line to the known result. This leads to an assumption that she does not know how to deal with an empty number line, and in this case fails to understand that the accumulative number on the entire jumps refer to the second number on the addition problems. However, I could argue as well that she understands the number relations between adjacent numbers, such as 39 + 10 = 49; 49 + 10 = 59; etc because counting by tens is not a problem for her.

Problem: In the competition for Indonesian Independence Day on the 17th of August, Annisa has to collect 50 hidden Indonesian flags. Annisa already has 19 flags with her. How many more flags does she have to find?* So, Annisa has to find one more flag



So, Annisa has to find 1 more flag

Figure 5.16. Annisa's written assessment for no.5 (21 August 08)

This result supports my previous assumption that she does not know how to deal with an empty number line as a model for mental calculation, in this case to understand that the accumulative number on the entire jumps refer to the second number on the addition problems, in this case 10 + 10 + 10 + 1 = 31. She knows where to end, i.e. 50 and where to start, i.e. 19, but she does not have the relation that 10 + 10 + 10 + 1 = 31 as the total number of flags that she has to find in the problem. It tells us that Annisa does not understand the context problem as well.

Answers and Recommendations

These observations and interpretations lead to "yes" as the answer to research question (1.3). This could happen because there was less attention on mental calculation than on the empty number line, which makes the empty number line the most important feature to solve problems. Moreover, a direct instruction from the teacher to use an empty number line is another reason in this case.

This suggests to us (teachers and researchers) that we must be thorough in teaching children to perform mental arithmetic strategies on an empty number line. All chosen activities, examples, and instructions should make sense to the children. Exploring more number relations in which a number is decomposed (for instance, 28 = 20 + 8 or 10 + 10 + 8), and talking more about the mental strategies, as on the number line, could be very helpful in preventing this problem, i.e. children interpret an empty number line as a newly taught calculation procedure. Moreover, providing context problems and problem strings might help to develop their understanding of number relations

5.4. Conclusions

5.4.1. Answers to the First Research Question

1. *How do Indonesian children who are used to perform an algorithm strategy develop their mental arithmetic strategies on an empty number line to solve addition problems up to 100?*

The development of children's mental arithmetic strategies is influenced very much by the previous knowledge of the children. Since Indonesian children are acquainted with the algorithm procedure, they tend to perform the algorithm strategy in explaining their answer. Although some of the children show their development in mental arithmetic strategies by means of the contextual situation and the empty number line, there are still some children who do not develop their understanding in constructing mental arithmetic strategies on an empty number line. They find difficulties in dealing with the 'empty number line' strategy, because the empty number line does not make sense for these children when they are able to solve addition problems using their number sense or the algorithm strategy.

1.1. How do children develop a framework of number relations to construe flexible mental arithmetic strategies?

The children develop a framework of number relations during the 'locating numbers on an empty number line' activity. The 'combinations that make ten' game also help them to build the framework of number relations. The discussion on sharing and comparing their strategies throughout the activity, they gradually develop their understanding of number relations. Rezon's understanding of number relations helps him in solving a problem using mental arithmetic strategies on an empty number line. Yona also shows a great performance in calculating mentally. The result of the experiment shows that Yona develops her number relations that could help her solving problems flexibly.

The results support the conjecture that a locating numbers activity could be used to explore number relations and to enhance children's counting strategies. By understanding number relations, children do not only operate numbers and find the correct answer, but they also realize what they are really doing in performing mental arithmetic strategies on an empty number line, and attempting to find a sophisticated and reasonable answer.

1.2. How flexibly do the children use mental arithmetic strategies on an empty number line to solve addition problems up to 100?

Yona and Rizqy's written final assessment shows their flexibility in doing mental arithmetic strategies on an empty number line. For example, Yona prefers the splitting strategy in solving 24 + 65 and start from 60 instead of doing common jumps of ten from the first number as the fixed number on an empty number line. She is able to represent her thinking in the splitting strategy on an empty number line, although it has never been taught in the classroom. Additionally, she is able to solve the problem 41 + 19 by performing a jump of 20 at once and a jump of one backward. In this case, she does not need to perform an algorithm strategy to solve this problem that requires a conversion in which 1 and 9 makes 10 that should be converted into one for the tens.

1.3. Do children interpret an empty number line as a newly taught calculation procedure?

The empty number line has emerged as a newly taught calculation procedure for some of the children. They call it as 'the number line' strategy in which they simply perform the similar jumps of ten for any kind of addition problems. The result of the written assessment and the children's performance in the classroom show that the children do the 'empty number line' strategy without an understanding. They perform jumps of ten strategy on an empty number line repeatedly on various problems without an understanding of the number relations involved; for example Annnisa does not understand that the accumulation numbers on entire jumps that she performs – 10, 10, 10, and 1 – is the number that is added to the first number. Moreover, in solving the 'bare missing add-end' problem, Yona uses the empty number line only to accomplish the instruction of this problem because she performs the jumps of ten similar to the other problems without understanding the meaning of each jump.

This could happen because there was less attention on mental calculation than on the empty number line, which makes the empty number line the most important feature to solve problems. Moreover, a direct instruction from the teacher to use an empty number line is another reason in this case. On top of that, there is lack of discussion about one's reasoning in solving certain problems using certain strategies. Therefore, the empty number line is not yet a model for the children's mathematical reasoning that they can use flexibly.

5.4.2. Answers to the Second Research Question

2. How does the emergence of an empty number line through a measurement context support the development of children's thinking in constructing number relations to support mental arithmetic strategies?

The empty number line emerges very naturally from the activity of recording a measurement result into a model *of* the measuring situation. However, the development of the empty number line from the 'model *of*' to the 'model *for*' is not established yet. For some children, the empty number line is not yet a model for their mathematical reasoning that they can use flexibly.

2.1. How does the measurement activity with a string of beads support the development of children's counting strategies and their number relations?

The proposed measuring activity doesn't help children to support their counting strategies and to come up with the intended patterns. The context of measuring children's wrists to make bracelets gives the children the idea to measure all round things on their body, such as the circumference of their waists and necks. As a result of these round things, children cannot immediately perceive the pattern of a string of beads, which could improve their counting strategies. Moreover, there is no need for the children to use measurement when they are arranging the beads, so, they create their string as beautiful as possible without considering the need for measuring.

2.2. How does a pattern of a string of beads support the development of children's counting strategies and number relations in their thinking?

The pattern of the string of beads encourages children's counting strategies to develop from counting by ones to counting by tens. Alternating the color by tens in a string of beads allows children to see the beads as chunks of ten beads, which leads to counting by tens. The pattern helps children to count by only looking at how many chunks of ten beads and how many more to the end of the measured object. This is in fact a very sophisticated strategy in counting mentally supported by the pattern.

2.3. How does an empty number line support the development of number relations in children's thinking?

The empty number line is proven as a powerful model to develop number relations in children's thinking. In the activity 'locating numbers on an empty number line', the children estimate the position of numbers using the known number. If 0 and 100 are the known numbers, the children will use their number relations to determine that 50 must be in the middle of 0 and 100.

Moreover, the number 20 must be closer to 0 than to 50, the number 90 must be closer to 100 than to 50, and so on.

Providing a string of beads on top of an empty number line in the activity also improves children's understanding of number relations. They use the pattern of the string of beads (the ten structure) to speed up their counting. Furthermore, they do not need to count from the beginning to locate a number, they can just look at the nearest number and count on/ back.

2.4. How does an empty number line support the development of mental arithmetic to solve addition problems up to 100?

The empty number line does support children's development of mental arithmetic strategies. Furthermore, the empty number line also serves as a model for the children's flexibility and their number relations in solving addition problems. The empty number line provides a visual representation of children's calculating strategy that can be perceived and discussed by the classroom community. In this case, the empty number line facilitates children's discussions in the classroom about different strategies. The empty number acts as a representation of children's thinking that could help them to trace and realize a mistake in a mental calculation strategy. However, the empty number line cannot support the development of children's mental arithmetic when the idea of number relations in calculating mentally is neglected in the classroom discussion.

5.4.3. Answers to the Third Research Question

3. How do the role of the teacher and the classroom culture support the development of children's thinking in constructing mental arithmetic strategies?

The teacher's beliefs about mathematics education in the classroom have an effect on the development of children's thinking in constructing mental arithmetic strategies. The teacher's beliefs are still influenced by traditional mathematics teaching, where the children are expected to give answers as the teacher wishes. Instead of giving the freedom to solve problems in the children's own way, the teacher gives a direct instruction to use the empty number line to solve problems that can limit children's thinking in doing mental arithmetic. This causes the empty number line to become a meaningless strategy for children. The teacher's role in developing a contextual situation also has an effect on the development of children's thinking while engaging in the activity.

However, the teacher helps children in their development by facilitating the discussion; providing tools (several strings of beads with different patterns); posing questions to enhance children's thinking; building a classroom culture in which children can participate in the classroom as validators; and encouraging children to contribute to the classroom discussion for the sake of the development of their learning process.

The open classroom culture (see appendix 4) which was attempted to be developed through the teaching experiment is not established yet. The children often forget their agreement and the teacher sometimes does not remind the children. Parts of the classroom culture, such as trying to make sense for our answer and reasoning; and be responsible in the classroom activity are not yet established. Therefore, there is little discussion about "how do you come to that answer?" or "do you understand of her/his thinking?". The culture of 'always obey the teacher's instruction without any reasons' is also another significant factor in the development of children's mental arithmetic strategies.

6. Discussion

In this design research study a sequence of instructional activities in the Hypothetical Learning Trajectory (HLT) was developed and implemented in the second grade of an elementary school in Jakarta, based on the Realistic Mathematics Education (RME) theory. The result of this study will contribute to the development of grounded instructional theories on addition up to one hundred using mental arithmetic strategies on an empty number line. The previous chapter provides the result of the design experiment together with its retrospective analysis to answer the research questions. In this section we will discuss the results in a broader sense and reflect on the theoretical framework that was outlined in the second chapter. Remarks and recommendations for teaching and for future research will be deliberated as well.

The Heuristics of Realistic Mathematics Education

In this section we reflect on the RME heuristics of *didactical phenomenology*, *guided reinvention*, and *the emergent modeling*. The didactical phenomenology heuristic has contributed to the design by providing a 'measuring context' as a meaningful contextual situation for the starting point of the children's progressive mathematization to construct mental arithmetic. Developing a contextual situation must be carried out very appropriately so that the children find it reasonable to engage in the activity. In this experiment, the crucial moment of 'developing the context' was not very successful. Although the teacher manual provided logical steps for developing the context, the teacher seemed much influenced by her previous experience in the measuring activity. At the beginning of the activity, children did not develop the need to measure when arranging the beads. Therefore, the children did not gain a mathematical reason for doing the activity. They simply made a beautiful pattern of the string of beads because they liked it.

It was crucial to study children's pre-knowledge in the domain before engaging in the teaching experiment. An interview at the end of the teaching experiment revealed children's conceptions about mental arithmetic strategies. Most of the children already had a basis for doing mental calculation, because they enroll an informal mathematical course, such as *kumon* and *finger math.* This should have been one of the didactical phenomenology considerations for choosing the proper starting point in the children's learning trajectory.

The children's progressive mathematization from reality (the measuring activity) to formal mathematics (adding numbers up to 100 using mental arithmetic strategies) process was enhanced by the use of a model (the empty number line) that emerged first from the contextual situation as a model *of* the measuring situation and developed to a model *for* mathematical reasoning in mental strategies. The empty number line emerged as a model *of* their measuring activity. The activity of recording their measurement results on an empty number line was a reasonable way for children to experience the emergence of the empty number line. However, the development of the empty number line to be a model *for* their mathematical thinking was not established yet. This was probably because the classroom mathematical discussion mostly focused on the use of the empty number line to record their solutions instead of on the mental strategies performed in children's heads. Moreover, the learning environment in the classroom suggested that children performed their calculation on the empty number line.

The misused problem strings by the teacher did not fulfill the aim of the problem strings as intended. The children did not discuss their mental strategies to solve problems (by explaining a solution verbally without the use of pen and paper while the problem strings were presented on the whiteboard one at a time), but they did discuss their strategies on an empty number line. For some children, an empty number line was a less sophisticated procedure to solve addition problems, because it only caused a new problem in dealing with the procedural 'jump-of-ten'. The unexpected result was that children performed an algorithm strategy to find a solution for an addition problem and then carried out the strategy which was instructed – the empty number line. They in fact performed the algorithm strategy first and then adjusted the jumps of ten with the earlier solution. They did not use the empty number line as a model *for* their reasoning, but as a calculation tool. However, the flexibility in children's performance while solving problems on an empty number line appeared in their written final assessment and in their performance in the classroom. Furthermore, a detailed analysis of Yona's thinking in her context of the missing add-end problem showed that the empty number line allows children to trace and manage mistakes in solving problems (Gravemeijer, 1994a).

The apparent success in the experiment was the activity of 'locating numbers on an empty number line' to explore number relations. For example, a sophisticated strategy in locating the number 92 on an empty number line under a paper-string of beads emerged after the classroom discussion. The children come to an agreement that locating 92 by counting back 8 times from 100 was better than counting on by tens from 10 to 90 and counting on two more. Then the children gradually developed vertically in their progressive mathematization process from the reality object of ‘counting the beads’ into the mathematical concept of number relations – the number 92 is 8 less than the number 100. In this respect, the empty number line could represent the counting number in a linear model and could bridge the gap between the ‘reality matter’ and ‘formal mathematics’.

In the progressive mathematization process, on the one hand children were given freedom, while on the other hand they needed to be guided carefully by the teacher. For example, children were given freedom to apply their counting strategies in determining the measurement result, and also to use various strategies in solving addition problems. On the other hand, the teacher should guide them by developing a contextual situation in a reasonable way, facilitating their discussion, posing questions to correct mistakes and to enhance children’s thinking, providing tools, and building a classroom culture in which children have a responsibility to contribute to the development of the learning process. It does take time to change and influence teachers and children’s beliefs about their own role in respect to the guided reinvention process. Although the classroom community involved in the experiment has started to build an open classroom culture, the culture has not been established yet. They need more time and more consistency in changing their beliefs and roles in the classroom mathematical practices. This explains why the teacher’s authority in the classroom somehow limited children’s freedom in engaging with the activity, for example, the children were given a direct instruction to solve every addition problem using a number line.

The Design Research Methodology

This design research study gains *ecological validity* in the sense that the design research helped in bridging the gap between theory and practice. A detailed description of the teaching learning process, the development of children’s learning process, the role of the teacher, and the emergence of mathematical practices (see chapter 4 & 5) provide a basis for adaptation to other situations. Moreover, an example of the teacher manual, a description of classroom culture, and the written assessment together with its rationale (see appendix) offer an idea of how to put theory into practice. Furthermore, to improve the *internal validity*, the researchers collaborated in interpreting and analyzing a fragment of a classroom situation or an individual learning process. One of the researchers sometimes plays the role of ‘devil’s advocate’, by doubting an analysis and an interpretation made of a specific situation. In this respect, the result provides a trustworthy justification and interpretation about children's thinking and the whole teaching learning process that leads to the correctness of the findings.

Our learning process in this design research could be traced back by investigating the motivational aspect and the problems arose (chapter 1), the chosen theoretical framework that guides the design (chapter 2), the procedures followed (chapter 3), the development of HLT (chapter 4 and appendix 5), and the analysis process with its framework together with the failures and successes in the experiment (chapter 5). Together with the inter-subjective agreement among the researchers, these suffice to meet the methodological norms of *trackability* and *internal reliability*.

Recommendations for Teaching-Learning Process and Future Research

There are two recommendations for future research. First, it might be better to conduct the design research in a classroom where the algorithm strategy has never been taught, so that the children might come up with their informal strategies. The second is about the significant role of the teacher. The negotiation process with the teacher to come to a mutual understanding about the teacher's role and the mathematics takes time. Therefore, regular meetings to negotiate the mathematics, the classroom culture, and the role of the teacher are much suggested.

In addition to the two recommendation points above, the points of recommendations below could also improve the teaching-learning process in a classroom and the future research:

1. Regarding the measurement context used in the teaching experiment, there was no child who came up with the expected patterns in arranging the beads. The interpretation and the analysis in chapter 5 suggest to start the measuring activity with measuring straight lengths instead of measuring round things. Moreover, to make children really experience the need of using patterns in measuring, the measuring activity should be repeated once more after a classroom discussion in which children compare different patterns and come to the conclusion that using a certain pattern can help them with counting. This will make all children get involved in experiencing counting the number of beads by tens (or by fives).
2. An empty number line did not develop from a model *of* a situation into a model *for* mathematical reasoning since there was less attention to the mental arithmetic strategies than to the use of the empty number line. For the next time, more

discussion on “How to solve the problem mentally?” instead of “How do you solve using an empty number line?” should be conducted by posing problem strings that ask for children’s mental strategies, and then sharing and discussing their strategies. Additionally, the teacher should avoid instructions or environments that suggest children to think that an empty number line is a newly taught procedure in doing calculations, such as giving direct instruction to use an empty number line. Furthermore, exploring number relations in which a number is decomposed (for instance, 28 = 20 + 8 or 10 + 10 + 8) could help children understand the mental process involved when dealing with the empty number line. All chosen activities, examples, and instructions should be meaningful for children.

3. Due to the fact that the children continuously depend on their algorithm strategy (see the analysis of Annisa), context problems that do not suggest the use of algorithm should be envisioned. There must be a reason for children to perform mental arithmetic strategies instead of pen and paper algorithm. For example, the context problem below may suggest children to perform the jump-of-ten strategy in finding the answer: “Sasya has 36 marbles in her pocket and her father gives her 10 marbles every week; how many marbles does Sasya have after a week? How many marbles does she have after two weeks?” and so on.
4. In relation to the classroom culture, giving more appreciation to children when they are doing different strategies, comparing the differences and finding the effective strategy should be made explicit. Moreover, the discussion process in which children give different strategies or compare strategies, should attempt to make sense of what happens in their thinking, not only to know how one solves certain problems with certain strategies, but also to explore why one finds the strategies.

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Appendix 1. An Example of the Teacher Manual

Unit 1
Measuring using a String of Beads
2 x 30 minutes (2 lessons)

Mathematical Learning Goals :

1. Children can measure things around them using a string of beads.
2. Children can apply certain patterns in their strings of beads to help them determine the number of beads while measuring.
3. Children can compare numbers through a measurement context.
4. Children can count by tens in determining a number of beads in the string of beads which has alternating colors by tens.

Planned Instructional Activities :

A. Developing the context
   - Beads as accessories such as necklaces and bracelets in daily life.
   - How many beads do you need to make a bracelet for you? (estimation); how many beads does your friend need? Who needs the biggest number of beads?
   - From comparing the number of beads needed to make a bracelet, children will have an idea that they can use the string of beads to find measurements of certain things around them.

B. Measuring for Celebrating the 63rd Indonesian Independence day
   - Children work in pairs in the measuring activity.
   - Children measure the circumference of their heads to make a white and red paper head band in celebrating Indonesian Independence Day.
   - Children measure other things around them.
   - The teacher and observers observe children's strategies in measuring and in determining the measurement results.

C. Preparing for the math congress
   - Making notes while children's engaging in the activity, such as children's strategies in counting, patterns that children use in counting, etc (this informs

Tools :

1. beads in two colors for each group
2. strings
3. children's worksheets
4. pen to write (because children below third grade are not allowed to bring a pen to school)

A. Developing the Context

- Context: measuring using a string of beads
- The teacher shows the children a beautiful string of beads with enthusiasm and starts a discussion in the classroom about the use of beads in their daily life.
- Some children might answer that beads can be used as accessories such as necklaces and bracelets. If so, ask children to make a bracelet for their partner in each working group so that every child has a responsibility to make a nice bracelet for their partner.

Before the children start to make a bracelet using beads, give a challenge for them to estimate the number of beads needed for a bracelet
* **This is intended to practice children's estimations in measurement context that could help develop their number sense.**

- Some patterns that might come out from the children's bracelets are: a bracelet without certain pattern, a bracelet in one color, a bracelet with alternating color (ones, twos, etc), or other patterns.

Look carefully at how children count the number of beads in their bracelets
Possible strategies are:

1. count by ones (1, 2, 3, etc)
2. count by twos (2, 4, 6, 8, 10 or 1, 2, 3, 4, 5, 10!)

The teacher's notes (observation results or comments on the activities):
For example:
How do most of the children count the number of beads?
Are there any strategies which are not mentioned here?

- Once the children know the number of beads needed to make a bracelet for their partner group, pose a question to them: "Does everyone need the same number of beads to make their bracelets?"

This question is intended to build the children's measurement sense and to develop their understanding in comparing numbers at the same time.

- The next question is: "Who has the biggest measurement for their wrists? Who has the smallest wrist?"

This question is leading to the measurement concept that could be a reason for children to do measuring activity using a string of beads.

B. Measuring for Celebrating the 63rd Indonesian Independence Day

- In celebrating the 63rd Indonesian Independence Day, children will make a headband for each of them from paper and in red and white color to symbolize our flags. The problem that arises is will they make the same measurement of headband for each child? Therefore, they are presented with a string of beads to find the result of measurements.
- The teacher shows the children three different measurements of headbands. The first one is too small, the second is too big, and the last one fits the teacher's head.
- The teacher ask the children: "How do you make a headband for every child so that it fits you?"
- The children are expected to suggest measuring using a string of beads, so that the next activity will emerge from the children's contribution.
- After all children agree on measuring using a string of beads, they are given worksheets and a pen.
- The teacher challenges the children to measure other things around them besides their heads

When the string of beads is too long, distract the children while they are counting so that they will think of another strategy which is better and more efficient in counting and in tracking the results. This is related to the string of beads patterns that could help them counting.

Observe children's counting strategies!
Some strategies that might come out:

1. count by ones (1, 2, 3, etc)
2. count by twos (2, 4, 6, 8, 10 or 1, 2, 3, 4, 5, 10!)
3. count by fives
4. count by tens, etc

- After children can measure certain things around them, lead a classroom discussion on their different patterns. The strings of beads are hanging in front of the classroom so that children can compare the patterns which could help them in better and faster counting.

Children who are already able to count by tens have a concept of ***unitizing,*** in which they understand a new unit, for example 10 that consists of ten ones. Thus, they can count by tens: 10, 20, 30, etc. It is not a simple thing for low level children to understand the unitizing concept, because what they know before is adding or subtracting one every time they count.

The emergence of a pattern in a string of beads with alternating colors in ten or fives could improve children's counting strategies. A classroom discussion where they share strategies and decide the effective way in counting, supports the development of children's learning process.

C. Preparing for the Math Congress

- Collect children's worksheets including their pictures of their string of beads patterns.
- Collect children's string of beads for a classroom discussion the next day.
- Observe children's strategies in counting.

Appendix 2. Data Generation

The core idea of a design research is the local instruction theory in which students' learning trajectories are designed and developed by first doing thought experiment about the level of students' thinking in a specific mathematical domain and formulating activities as well as designing means to support the development in students' thinking processes. In a design research, data are very important regarding their goals in the improvement (five-tuning) of the local instruction theory (LIC) and also as evidences of your theory. In order to reach those goals, the selection of data should be made clear.

Data that are going to be collected in improving the LIC are about the learning processes of students, and also the role of the teacher. When we discuss about learning processes, is it possible to see learning processes in students' mind? What is learning itself? And how to study learning? Since learning processes is a very abstract activity in students' mind, we could never be able to see the whole learning process in students' minds directly. Moreover mathematics is about concepts, thought, ideas, and strategies which are very abstracts. What we could do is trying to know students' thinking processes in dealing with situations or in solving problems by asking them to explain what their thinking is. If somebody has learned something, there must be a change that we could observe. As an example, a scrap paper for writing down strategies and thinking in solving problems could be helpful to obtain information about thinking process of a student or how a student has come to an answer.

Three basic forms of data-collection are (1) collecting and analyzing documents; (2) making observations about what people do and say; (3) asking questions about what people think and perceive (opinions, attitudes) or what people know or can (knowledge, strategies) in the form of interviews, questionnaires, or tests. In observations, we can only see the actual behavior of ones acts (nonverbal behavior) and words (verbal behavior) without knowing ones thoughts or opinions. To observe a classroom situation we should already have had a framework in order to focus on important and critical moments that we have to look at. Therefore, the goals of our research should provide framework in our observations. In observing, we are not only looking but also seeing and not only listening but also hearing. For example, in a lesson observation, it is important to make video / audio registrations and notes of whole class discussions as well as the group works.

As a researcher, we should decide what types of observation that we are going to use, is it participating or not; structured or unstructured; open or hidden. When we decide to be a participating observer, we could have interactions with students and the teacher, we could also engage in the learning-teaching experiment by taking over the class or supporting the teacher. Make sure that the teacher is well informed about our role as observers. In unstructured observation, the instrument that is used is the observers themselves with their knowledge and experiences. The advantage of collecting data by observations is that we could have rich data. However, the disadvantages are (1) no opinions and motivations; (2) selection of relevant observations, (3) interpretation of the result of observation. Therefore, interpretative framework is applied in the process of selection and interpretation.

The second method of collecting data is interviewing, which could be conducted by mini interviews, big interviews, questionnaires, or tests. There are steps to prepare a success interview: formulate the aim of the interview – make a list of topics – design interview questions – formulate an introduction – do a pilot interview – and redesign the interview questions. Interviews with students as well as with the teacher should be conducted not only in the beginning and in the end of the teaching experiments, but also in between. Mini interviews with a group of students could be conducted in the beginning to assess their prior knowledge and their thinking processes, in between to understand their ideas, strategies and struggles in constructing a mathematical idea, and also in the end to assess their knowledge and the changes after the learning processes. In interviewing the teacher, we could gain as much information as possible before the teaching experiment about their views, perceptions, and attitudes in teaching math. (Ex; “What is your view on learning mathematics?”; “What is your role as a teacher in students’ learning?”; “What is the function of models and tools?”; etc). We could also ask the teacher to make a concept mapping about one topic that we are interested in. In between the teaching experiments, stimulated recall interviews in which both the researcher and the teacher do observation of a classroom situation in a video, is conducted to know the teacher’s thinking (Ex. What happened there? What did you think? What did you do? Etc).

On top of all the methods of collecting data, all the data have to be coded: name school and teacher, class, subject, date, length of time, etc. Documents should be copied and listed, Observations in video/audio recordings and in notes should be coded, Interviews should be recorded and noted, and students’ written works should be copied and named.

Appendix 3. Assessment

In this report, written assessment sheets for children at second grade will be presented, not only assessment after but also before the teaching experiment. Streefland (1981) has argued that assessment should be viewed not in the narrow sense of determining what the students has learned but from the standpoint of educational development-that it should provide teachers with information about what to teach (in Fosnot, Dolk, 2001). In the case of my design research, the assessment before the teaching experiment will give information about how children manage addition and subtraction problems up to 100, and moreover how children manage context problems in their daily life. This information will help us to refine the conjectured local instruction theory and to give us more anticipations of children's thinking. The assessment during the teaching experiment will give information about children's developmental thinking to perform the ongoing analysis of the conjectured local instruction theory. Furthermore, the assessment after the teaching experiment will provide information about children's interpretation of an empty number line and how flexible children to manage addition and subtraction problems both in a bare number format and in context using mental arithmetic strategies.

Van den Heuvel-Panhuizen (1996) suggests several criteria for assessment to capture genuine mathematizing:

1. Students' own mathematical activity must be captured on the paper.
2. The test items must be meaningful and linked to reality.
3. Several levels of mathematizing must be possible for each item.
4. Assessment should inform teaching.

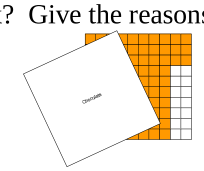
Based on above criteria and the mathematical goal of my design research, below are the written assessment sheets for children.

A. Assessment before the teaching experiment

Addition and Subtraction for Numbers up to 100

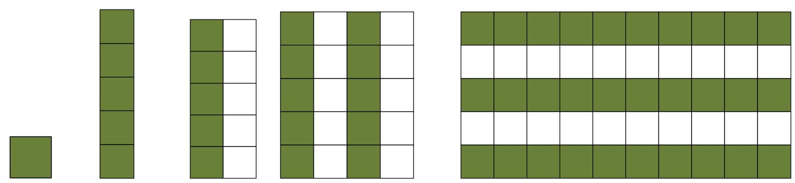
Solve the problems below by describing how you come to the answer!

1. Look at the chocolate box below!
   Bu Amy is a chocolate seller. She is preparing an order for 100 chocolates in a box.
   How many chocolates are in the box? Give the reasons for your answers!



2. Look at the picture below!
   Have you ever seen how stamps are sold in a post office?
   Picture (i) shows a picture of one stamp
   Picture (ii) shows a picture of a piece of 5 stamps
   Picture (iii) shows a picture of a piece of 10 stamps
   Picture (iv) shows a picture of a piece of 20 stamps
   Picture (v) shows a picture of a piece of 50 stamps

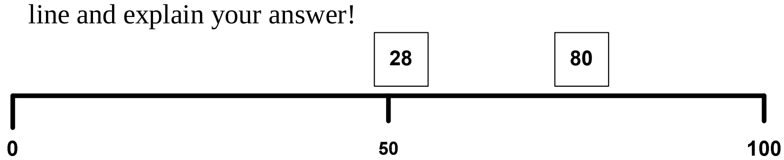
   Can you determine which pieces of stamps that you want to choose for:
   (a) 37 stamps
   (b) 65 stamps
   (c) 14 stamps



3. If there are 53 beads in a jar, and you need 45 beads to make a necklace, how many beads are left in the jar? Explain your answer!

4. Lisa is 3 cm taller than Boby. If Lisa's height is 97 cm, how about Boby's height? Explain your answer!

5. Do you know the position of the numbers below? Place the two numbers on the number line and explain your answer!



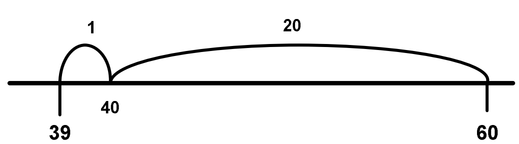
6. Circle the bigger number in each box below!
   Then add the number to its partner! (Use as many strategy as you can)

   28 81 | 46 71 | 62 38 | 22 31

An example:

39 21

A strategy:



39 + 21 = 60

*The real assessment sheet for children is arranged with empty space between two problems to provide a space for children's answers, and also in a bigger font size.

The pre-assessment were intended to investigate children's pre-knowledge in dealing with numbers and its operations in order to adjust the starting point and the activities in the HLT. The pre-assessment inform the researcher not only about children's knowledge in certain aspects, but also give experience for the researcher in designing problems using proper language for children.

The first problem was intended to find out children's counting strategies in determining amount. Children might count by ones or using the structure of ten (the structure of the box) to count by tens. However, the picture of chocolate box is designed in such a way to encourage children's thinking in counting using the structure of the box. The result shows that most of the children didn't understand the context of the problem properly. They didn't get involve to the problem because they never experienced that kind of situation. It suggests us this kind of context problem must be preceded by an experientially real situation for children. However, there were also children who show their understanding in dealing with the problem. Some of them counted by ones by imagining the chocolate under the cover, some of them counted by tens using the structure of the box, and there was also a number of children who subtract the total by the empty box.

The second problem brought about children's confusion again. First, the children have never been experienced with stamps. So, they even don't know what stamp is and what it is for. Second, the picture of the 'one' stamp, 'one' piece of 5 stamps, 'one' piece of 10 stamps, and so on gives an idea that the children only have that number of stamps illustrated in the figure. Therefore, it is impossible to have 2 piece of 'one' stamp. Although this problem is intended to develop children's counting strategies using the structure of tens and fives, the children can not grasp the idea because the context is not meaningful for them.

The third problem deals with subtraction. Children are expected to use their informal strategies, but most of them use the algorithm strategy. As a result, children who have inadequate understanding of solving subtraction problems using algorithm made common mistakes, such as: treat 3 – 5 similar to 5 – 3 and forget to subtract one tens from the 5 of 53. However, there was also a child who performed her informal strategy by drawing the beads and crossing the beads that she used for making a necklace.

The fourth problem still deals with the third problem. The problem was expected to give an idea for children in dealing the subtraction problem as 'taking away'. While the third problem deals with subtraction as 'adding on'. Most of the children perform the algorithm strategy to solve this problem, although a small number of children can not solve this problem. Moreover, there were children who still do not understand what 'cm' means, because they have never been taught.

In the fifth problem, some children locate the numbers using their number relations. They use their number sense and their number relations that the number 80 must be on the right side of 50 and nearer to 100 than to 50, and the 28 must be on the left of 50 and nearer to 50 than to 0. Some of them used their estimation and gave a reasonable answer. But for some of them who count by ones from the beginning, they placed the number on the wrong pace because they did not realize the number relations in it.

The last problem was realized as a mistake from the researcher, because an example of solving the problem was provided. The example should not have been put on it, because it caused confusion to the children. The strategy on the empty number line was not meaningful at all, because they have never had an experience with an empty number line. Giving an example of solving the problem on an empty number line limited children's strategies.

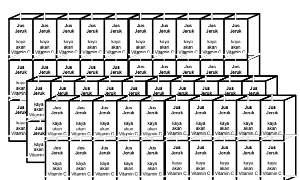
On top of all, most of the children have difficulties in explaining and justifying their answers. They get used to only give an answer without explanation and reasons. The children need to learn more about explaining their answer by exercise in the classroom and by a classroom discussion

B. Assessment after the teaching experiment

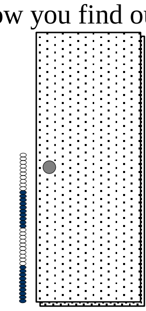
Thursday, 21 August 2008

Solve the problems by giving your explanations and reasons!

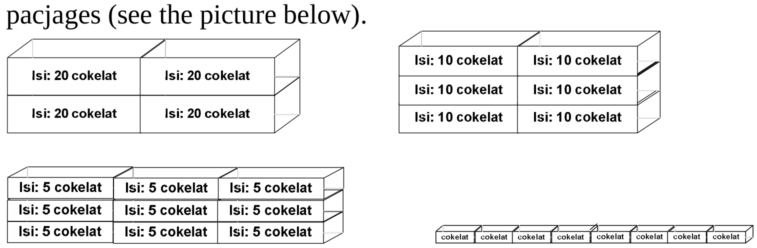
1. Jenna is going to hold her 7th birthday. Jenna will invite 65 people to her party. Do you think the number of juices in the picture below is enough for 65 people?
   Explain your answer!



2. How many beads do you think the height of the door in the pictures?
   Explain how you find out the height of the door!



3. Yona goes to a market to buy 37 chocolate. Yona finds several kinds of chocolate pacjages (see the picture below).



Which packages should Yona take to make her easier to carry them? Explain your answer!

4. Make couple of cards that have a sum of 28!

20 25 14

10 8

14 3 18

5. In a children's competition for celebrating the Indonesian Independence day on the 17^{th} of August, Annisa and friends have to compete to find 50 small Indonesian flags which were hidden. Annisa has found 19 flags, how many more flags that she has to find?

6. How do you solve the problems below using an empty number line?
   a. 41 + 19 = … b. 24 + 65 = ...
   c. 56 + 37 = … d. 39 + 48 = ...
   e. 34 + … = 61

Although at the beginning the design research will encompass addition and subtraction up to 100, at the end there was not enough time to comprise the subtraction problems. The missing add-end problem was the last subject that was discussed in the classroom. The first problem was intended to assess children's development in their counting strategies. The results show that most of the children develop their thinking after the teaching experiment. They can determine the number of juice in the picture using structure of ten. Although there was a small number of children who still counted by ones, the structure of ten in the picture encourage some of the children to count by tens.

The second problem deals with measurement situation in which the children have to estimate the height of the door using the string of beads available in the picture. This problem assesses children's performances in the measurement activity. At first, they were expected to use the 20 beads to estimate the total height of the door. However, the result shows that only a small number of children who used the beads (the pattern) to count by ten or by twenty. Some of them drew other beads (either in a similar size or in a different size) from the middle to the top, and some of them use a line to represent ten beads in a string.

The third problem is similar to the second problem in the pre-assessment. The context deals with chocolate which is experientially real for children. Their performance in this problem is much better than the previous one. Some of the children can give different combinations of the packages, although they sometimes did not recognize the need of the effectiveness in carrying the chocolate. This problem leads to the concept of decomposing numbers. For example, 37 can be decomposed into 10, 10, 10, and 7, or 30 and 7, or 20, 10, and 7, or 30, 5, and 2. It was expected that if children can deal with this problem very well, then they could be flexible in solving problems mentally.

The fourth problem seemed to be the easiest part in the whole assessment. The children can easily find combinations that make 28 from two numbers. This problem was also intended to develop children's flexibility in perceiving a number, i.e. a number can be decomposed in a several way. They could use this idea to shorten their calculating strategies in their head depending on the problems. For example, for the problem: 57 + 28, children can decompose 28 into 25 and 3 so that the 3 can be added to the 57 to get to 60, and they only have to add 25 more to the 60 to solve the problem.

The fifth problem is a contextual missing add-end problem in which children were expected to add on from 19 to 50. Some of the children solve this problem using the algorithm, but some of them using an empty number line. Although some of them can draw an empty number line that represents the situation, they do not understand the entire number on top of their jumps is the answer for the problem. The children who have good number relations are proven to be able to solve this problem correctly.

In the last problem, the numbers were intentionally chosen to observe children's flexibility in solving problems. Certain problems require different strategies for the sake of efficiency. The direct instruction of using an empty number line urges children to draw an empty number line for each problem although they do not understand how to deal with it. Some of them solve the problems by performing an algorithm first then adjust the jumps on the empty number line based on the result obtained by the algorithm. Nevertheless, some of them show their flexibility in solving the problems on an empty number line. Therefore, the empty number line must be introduced thoughtfully with a reasonable motivation of solving problems using different strategies than algorithm.

Appendix 4. The Role of the Teacher and the Classroom Culture

The Role of the Teacher

Due to the important role of the teacher in supporting the 'guided reinvention' process in children's learning, it is important that the role of the teacher is negotiated before the start of the teaching experiment. Below is a paper written by the researcher to give an idea about the role of the teacher. The teacher should reflect on the explanation given to her common practice in a classroom. This paper serves as a basis for the negotiation process between the researcher and the collaborating teacher.

The Role of the Teacher in Realistic Mathematics Education *

Puspita Sari, Freudenthal Institute, Utrecht University, the Netherlands

Realistic Mathematics Education (RME) is a reformed mathematics education during the second half of the 20^{th} century that was first developed by Hans Freudenthal, a mathematician in the Netherlands. Hans Freudenthal thinks that mathematics is not supposed to be considered from formal mathematics without meaning. Freudenthal argues that mathematical learning should start meaningfully from daily activities, that is '*Mathematics as a human activity*'.

One of the RME heuristics is '*guided reinvention*' in which students are given opportunities to reinvent mathematics in their own way. This brings a different and unique learning path for every student. This is of course a difficult task and challenge for the mathematics educators, because in a classroom with different kinds of students, a teacher has to be able to guide them so that every student will develop their learning process meaningfully. Therefore, a big question for us is "How could teachers guide their students so that every student can reinvent mathematics in their own learning path?" Answering this question is not as easy as we think, because *guided reinvention* means a balance between students' freedom in the reinvention process and the teacher's guidance to enhance students' thinking. The paragraphs below answer our big question through an illustration of a classroom situation in a primary school in the USA that applies RME.

Since mathematics is a human activity, students should start their reinvention process from a reality. Therefore, the first guidance from the teacher is **developing a meaningfully contextual situation** for students. A teacher of a primary school, Hildy, who has about 20 students in her classroom uses a 'measurement' context as a starting point of the emergence of an empty number line that could be used as a model for reasoning in solving addition problems up to 100 using mental arithmetic strategies. A standard ruler is not needed for the measuring activity, instead a chain of unit cubes is used as a tool. In determining measurement results using the chain of cubes, students can count the number of cubes. At the low level, students might count the number of cubes by ones, while at the higher level, they might count by tens to be more efficient. The possibility of applying different strategies for a context problems shows that the 'measurement' context is a rich contextual situation in which students are allowed to use their own strategies in solving problems.

* prepared as a guidance for the collaborated teacher in implementing the Realistic Mathematics Education in the *Design Research on Mental Arithmetic Strategies for Addition Up to 100* , Juli-Ags 2008.

Providing cubes in different colors as measuring tools is another role of the teacher in enhancing students' learning process. With two different colors, students can make their counting more efficient by employing a structure in the pattern of the chain, for example 10 blue cubes alternates with 10 red cubes. The pattern can help students to count faster by counting by tens. **Giving freedom to students in solving problems** is another guidance of the teacher. If the teacher uses too much intervention and instructions, students become less creative in solving problems. For example, students are given the freedom to create a chain of cubes. Students are expected to learn from each other and discuss with one another to come to an agreement of more sophisticated strategy in solving problems.

The next guidance is **anticipating students' strategies.** Why do teachers need to anticipate students' strategies? This is because the freedom of students allows various strategies to come up. Therefore, teachers need to anticipate students' strategies to envision the next learning trajectory for students. For example, when Hildy poses the problem 34 + 17 , what strategies might come out from students besides formal calculation (the algorithm)? Below are some strategies that might come up when solving the problem:

(i) 34 + 7 = 34, 35(1), 36(2), 37(3), 38(4), 39(5), 40(6), 41(7) (*counting on one by one*).
In this case, students perform *dual counting*, that is perform two counting sequence at the same time. When adding 1 to 34, students get 35, and so on.

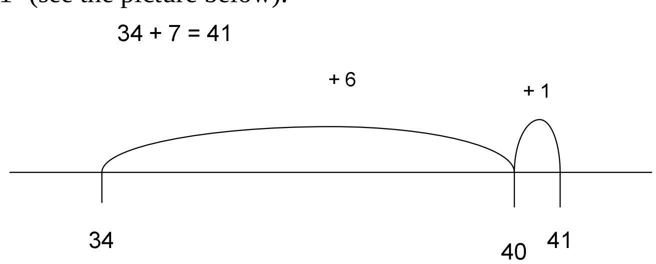
(ii) 34 + 7 = (34 + 6) + 1 = 40 + 1 = 41 ('jump-via-ten' strategy, to the landmark number, '40')
In this strategy, students understand the concept of combinations that make ten, that is 4 and 6 make ten. Therefore, students come to 40 and add 1 more to get the answer.

(iii) 34 + 7 = (34+10) – 3 = 44 – 3 = 41 ('jump-of-ten' strategy)
that is a jump of ten from 34 to 44, and then subtract the 3.

(iv) 34 + 7 = (4 + 7) + 30 = 11 + 30 = 30 + 10 + 1 = 41 (splitting strategy)
That is split the tens and ones and calculate them separately.

(v) Etc

The other guidance of Hildy in her classroom is **representing students' strategies on a whiteboard**. For example, when a student explains a solution of the problem 34+7 mentally with the 'jump-via-ten' strategy (ii), Hildy draws an empty number line and writes down the number 34, then she represents the jump of ten by the curve line as a representation of 'adding 6', then the smaller jump represent 'adding 1' to get '41' (see the picture below).



In representing students' strategies on the board, other students can see, learn, and understand the strategy, so the strategy becomes a topic of discussion in the classroom. Also, an empty number line allows students to trace and manage their mistakes, because it gives a visual representation of one's mental strategies.

It is unavoidable that students make mistakes in the process of their mathematical learning. A good teacher will never say "You are wrong!" to students who make a mistake. This could cause students to lose confidence. One possible way to **manage students' mistakes** is by posing challenging questions. **Posing important questions** to the students not only can manage mistakes, but can also **emphasize important mathematical ideas** that should be a point of discussion. Moreover, posing questions could also **raise students' thinking to the next level.**

A classroom discussion can determine the success of students' learning process in their classroom community. Since they communicate one another and share strategies, they could find a sophisticated way in solving problems. In this respect, the teacher should **facilitate students' discussions.** The teacher could facilitate students to have a pair discussion with their partner, so that every student has an opportunity to discuss with their partner in solving problems. When there is a student who comes up with an idea, it is for the teacher to ask the student to **repeat their thinking,** so that every student will be responsible for each other's thinking. **Supporting student to compare more effective strategies** is also another kind of facilitating students' discussions.

Another crucial guidance from the teacher is **developing a classroom culture** in a classroom community. A good classroom culture involves being responsible for one another's reasoning and thinking, so that students will listen and learn from each other. The culture of respecting each other should also be established in the classroom, so that the students will support each other in the development of their mathematical learning. The ***classroom culture*** should always be maintained to reach an open classroom culture.

Observing and evaluating students' learning development is another crucial point that a teacher must consider during the teaching learning process. By observing and evaluating, the teacher can trace students' development and anticipate their thinking, which might also inform a new better design for students' learning trajectory.

The Classroom Culture

The classroom culture as one of the means that support children's learning process should be established and developed to move towards a better classroom community. Before the teaching experiment was conducted, the researcher observed several classroom situations to provide information about the currently established classroom culture. Since the researcher worked with teachers who collaborate in an RME project, the classroom community seems to have started to build an open classroom culture where children are allowed to make mistakes, children have authority to make a decision in the classroom, and children are encouraged to share their thinking. Moreover, the children become validators whether an answer is correct or not, they respect one another by giving applause for the correct answer.

However, there are still some points that need to be established during the teaching experiment. Although changing a classroom culture was not an easy matter, the researcher tried to find a way so that the classroom community could always maintain the new classroom culture in their classroom. A poster that provides pictures and short messages arranged in a story lay out was expected to support the development of the new classroom culture. By providing a poster, the classroom community could always refer back to the new culture when they forgot, or acted out of the culture. The messages put in the poster were adjusted from the observed current classroom culture.

Pay attention to your friend or to your teacher when they are explaining. *Perhatikan guru dan temanmu yang sedang menjelaskan.*
Give your answer when your teacher allows you to do so. *Utarakan jawabanmu jika namamu disebut.*
Cooperate and learn from one another. *Saling belajar dan bekerja sama dengan baik.*
✓ a respectful and suggested attitude ✗ a less respectful and not suggested attitude

The points outlined in the poster were formulated very carefully to avoid children's misunderstanding and confusion. The first point is about responsibility in their classroom. The observed classroom situation showed that the children often did not pay attention to other students' arguments or ideas or to the teacher's explanation. This confined the classroom discussion to a situation where the children cannot reach a *taken-as-shared* argument for a problem or situation.

The second point deals with the children's discipline in the classroom. They often shouted together when answering the teacher's question. This makes the classroom very noisy, and the teacher cannot distinguish their different thinking for a certain problem.

The third point is expected to establish a new classroom culture where the classroom community has a responsibility to support one another. Through good cooperation in the classroom community and a good responsibility to learn from one another, the students are expected to come to a better learning process.

The result of the teaching experiment shows that the new classroom culture has not been established yet. The development of the new classroom culture requires maintenance from the classroom community, including the teacher who has the authority in introducing the new classroom culture. It also takes time to change teachers and children's beliefs about their own role in the classroom.

Appendix 5. The Development of the HLT

A. An Overview of the First HLT (March – April, 2008)

Mathematical Learning Goals

1. Children will develop a framework of number relations to construe flexible mental arithmetic strategies.
2. Children will be flexible in solving addition and subtraction problems up to 100 both in context and in a bare number format using mental arithmetic strategies

Starting Points

The Contextual Situation: Taking Inventory on "Celebrating the 63rd Indonesian Independence Day"

Designed Activities

Activity 1: Taking Inventory (Grouping, Unitizing, Combination that make ten)
Activity 2: Math Congress on Grouping by Ten and Unitizing
Activity 3: Practicing Addition and Subtraction Up to 20 (Constant Difference)
Activity 4: Measuring with a String of Beads
Activity 5: Modeling the Contextual Situation on an Empty Number Line
Activity 6: Locating Numbers and Exploring Number Relations
Activity 7: An Empty Number Line as A Model for Reasoning in Solving Addition and Subtraction Problems Up to 100 using Mental Arithmetic Strategies
Activity 8: Further Elaboration on Mental Arithmetic Strategies for Addition and Subtraction Up to 100

Hypotheses of Children's Learning Process

1. After engaging in a game of *combinations that make ten,* it is expected that the children will acquire the idea of '*combinations that make ten'* by understanding that 10 can be split into 1 and 9, 2 and 8, etc. After doing this activity, the teacher surprises children by carrying a big box of small flags into the classroom. The teacher tells children that they have to help the teacher in organizing the flags. The headmaster ask the teacher that he wants to challenge the second graders to organize the flags in a good way, so that it will be easier to divide the flags into every classroom which have quite different number of students (sometimes 38, 40, or 41, etc). The classroom community should discuss the plan of organizing the flags. It is expected that ***grouping*** the flags into ten will be the decision of the classroom community. The teacher emphasizes that children also have to record the number of flags on a piece of paper, so that they can make a good report to the headmaster. At the end of the inventory, a T-chart is provided by the teacher on a big piece of paper on the blackboard. A big piece of paper is used instead of the blackboard because the data should be kept, while if the data is on the blackboard, someone may clean the board. Organizing things in bundles of ten will help children to develop the idea of ***unitizing*** where they recognize that one whole group consists of ten parts, hence they will understand the place value, that 28 is two tens and eight ones. In flexible mental strategies, children are expected to apply unitizing in doing jumps of ten.
2. The start of the second lesson is reflecting on the first lesson in which children decided to make bundles of ten. Make children reason that by making bundles of ten it will be easier for them to count; instead of counting one by one, children can count by tens. For example, when they have to provide 38 flags for another classroom, they can just take 3 bundles of ten and 8 ones. Let children realize that organizing things in ten is useful and effective. In the second lesson, attention is paid to the chart instead of the flags. The object of discussion is the number of flags, i.e. how many bundles of ten, how many single flags, and how

many in total. Based on previous activities in '***combinations that make ten'***, children are expected to be able to find how many more they will need to make another bundle of ten from the single flags. This will help children in doing flexible mental arithmetic strategies when using the strategy '***jumps via ten'***. For example, they could see that 38 + 13 as 38 + 2 + 11 in order to make 40 from 38.

3. This activity is a way to automatize addition and subtraction up to 20, because at the end of grade 1, children are expected to be able to deal with subtraction and addition problems up to 20. This activity could be used when children in grade 2 still have difficulties to solve addition and subtraction problems up to 20. With this problem, children can recognize that to have a sum 0f 16 for example, they could add 1 and 15, 2 and 14, 3 and 13, and so on. This helps children to develop a framework of number relations in solving addition and subtraction problems up to 100 using mental arithmetic strategies. Why is it helpful? For example, in solving 16 + 9 they could think of 15 + 1 + 9 = 15 + 10 = 25. For the subtraction problems up to 20, children can recognize that the number 4 is the difference between 20 and 16, 19 and 15, 18 and 14, and so on. This fact will construct the idea that the difference between two numbers is the same when the numbers are added or subtracted to or from the same number. For example, to solve 51 – 16, children could think that the difference between 51 and 16 is the same as 50 – 15. Hence, the problem becomes 50 – 15 and much easier to solve by mental arithmetic strategies. Therefore, it is important to do this activity with children and discuss an idea of constant difference (subtraction as the "difference between two numbers" rather than as "take away"). Bring them to an understanding that to keep a difference between two numbers constant we have to add (or remove) the *same amount* to both numbers
4. The hypotheses of the fourth, fifth, sixth, seventh, and eight activities are more or less similar to the second HLT (in chapter 4)

B. The Try Out Activities (26 June 2008)

1. Introduction
2. Establishing a Classroom Culture
3. Activity with magnetic counters on the whiteboard (combinations that make ten)
4. Activity with a game (automatizing combinations that make ten)
5. Activity with number cards up to 20 (addition up to 20 and combinations that make twenty)
6. Activity with a number of colorful magnetic counters on the whiteboard (organizing/ structuring/ the use of tens)

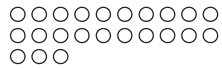
1. The introduction took about 5 minutes in which the children introduced themselves to me. In the introduction, I asked the children how many children were present in the classroom. Asking the number of children in the classroom is a natural way to start a lesson. Then, suddenly I started to think that I could gain more information about this question. Children shout together mentioning 21 and 22 as the result of their counting to determine the number of children present that day. I asked them how they know that there were 21 children present, or how they count the number of children present. I asked one of the children (Farel) to explain his thinking in obtaining 21 as the answer. Almost all of the children would want to express their ideas, but it seemed that they did not listen and pay attention to the children who were speaking. I repeatedly asked the children whether they have another way to get the solution or not, therefore I could explore some strategies from the children which are :

a. Counting one by one
b. Counting by twos or by fours (four in the first row, plus four in the second row makes eight, and so on)
c. Counting by tens (ten in a column and another ten in the next column makes twenty, plus one child makes twenty one.
d. Multiply four columns and five rows of children to get 20 and add one child more makes 21.
e. Counting the number of children who were absent and subtract that number from the total number of children. In this case, there were many strategies that children used to get the number of children who were absent. Some strategies are counting one by one; adding 7 and 8 and then add one more to get 16; doubling eight and eight to get 16.

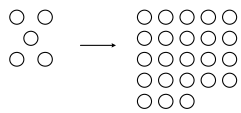
From those strategies, we could conclude that most of the children were already acquainted with doubling, counting by tens, counting by twos, multiplying, etc. This informs me that for the next teaching experiment, I could start activities that challenge children to solve problems using structure of tens. Moreover, almost all children did the same as Radya who used counting by tens.

2. I started introducing the classroom culture from the poster (see appendix 4) after a lot of noise was made by the children when they wanted to utter their answers together without listening one another. From my observation, the poster could help children to understand more about their role and their contribution in the classroom. It took about 3 minutes to introduce the classroom culture. All children agreed about the new classroom culture.
3. First, a ten frame was drawn on the white board. Then, I showed children a pack of five black counters, and a pack of five blue counters. I put the counters one by one in the ten-frame while children were counting the number of counters placed in the ten-frame. Then I showed children that I took out a number of counters from the ten-frame and asked them how many counters were inside the ten-frame and how many were outside. After that, I asked children to make drawings that give a picture of the ten-frame with a number of counters inside the ten-frame and another number of counters outside. They could decide by themselves about the number of counters that is outside and inside the ten-frame. All combinations that make ten occurred from children's drawings.
4. This activity was followed by a game in which children have to mention as fast as possible a number, the sum of which is 10 together with a number that I mentioned. Children enjoyed following this game. I think that this game could be a very powerful activity to make children remember combinations that make ten. At the end of this activity, children are expected to be able to automatize a couple of numbers that make ten. These combinations that make ten such as 1 and 9, 2 and 8, etc are very useful to do mental arithmetic strategies when they use landmark numbers like 10, 20, 30, etc. For example, in solving $34 + 17 = (34+6) + 11 = 40 + 11 = 51$.
5. A pack of number card (number 0-20) was given to each child to play a game. They were asked to arrange the number cards from 1 to 20 on their table so that they could show a number card that was equal to the number that I mention. The structure of ten helped Maudy to be able to the show numbers 3 and 13, 4 and 14, 7 and 17 consecutively. Yona is one of the highest level students. She prefers to use the structure of five in structuring the number cards. The next game was practicing combinations that make twenty. Keiza and Rara who were in the average level arranged combinations that make twenty. I gave children an opportunity to use number cards to help them to memorize combinations that make twenty and to follow the game. Children at a high level do not use the number cards anymore, they already remember combinations that make twenty.

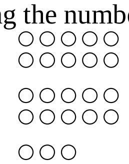
6. In this activity, I put 23 colorful magnetic counters on the whiteboard without structure and let children determine the number of counters. I ask one child to arrange the magnetic counters on the blackboard so that they will be able to count the number of counters easier. Farel showed the structure of ten in organizing the magnetic counters.



Then I asked other children who have other solutions in arranging the magnetic counters. Jenna moved forward and showed us structure of five. She was going to use the structure of dice (structure of five), but she decided to make a row of five and other rows of five with another row of three because she did not feel confident with the structure of dice. It is very interesting that the children in class 1C were already familiar with the culture of making decision together about the more efficient strategy. The children decided that the structure of ten is easier than the structure of five.



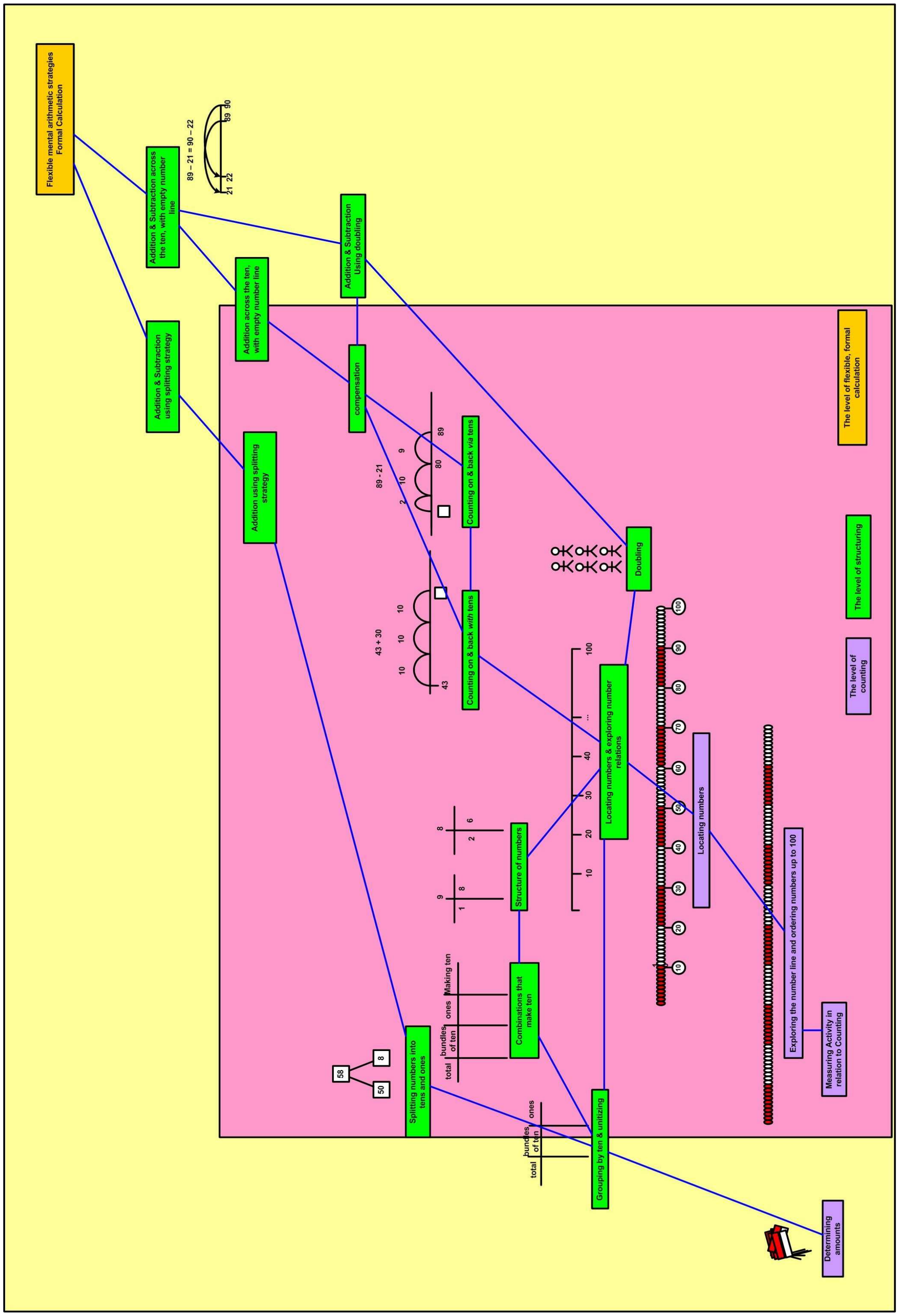
After that, Shafa came upwith another solution. She separated two rows of fives to make ten and another two rows of fives to make another ten. She thinks that this structure below is easier than the structure of five that Jenna made. However, at the end all children agreed that the structure of ten is the most efficient strategy in determining the number of counters easily.



C. **The Development of the Second HLT** (July – August 2008)

The try out activities, interviews, and other observations served as a basis for the development of the second HLT that was implemented in the teaching experiment phase. Information gathered revised the starting point of the HLT. The children involved in the teaching experiment show their good understanding of the ten-structure and unitizing. They do not need the activity of organizing and structuring as it was proposed at the beginning. Therefore, the second HLT offers a new starting point for the teaching experiment that is the measuring activity.

Moreover, due to the limitation of time, the teaching experiment did not cover subtraction problems up to one hundred. As a result, the second HLT provides a sequence of activities and conjectures about the children's learning process in addition up to one hundred using mental arithmetic strategies on an empty number line. The picture of the HLT (on the next page) gives a diagrammatic development in the HLT.


Flexible mental arithmetic strategies Formal Calculation
Addition & Subtraction across the ten, with empty number line
89 – 21 = 90 – 22
Addition & Subtraction Using doubling
Addition & Subtraction using splitting strategy
Addition across the ten, with empty number line
compensation
Addition using splitting strategy
89 - 21
Counting on & back via tens
43 + 30
Counting on & back with tens
Doubling
Locating numbers & exploring number relations
Structure of numbers
Locating numbers
Exploring the number line and ordering numbers up to 100
Combinations that make ten
Splitting numbers into tens and ones
Grouping by ten & unitizing
Measuring Activity in relation to Counting
Determining amounts
The level of flexible, formal calculation
The level of structuring
The level of counting

HLT 1 **HLT 2**