Supporting Children's Counting Ability as a Part of the Development of Number Sense with Structuring: A Design Research on Number Sense[1]

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1. Introduction

"Seven. What is seven? Seven children; seven ideas; seven times in a row; seventh grade; a lucky roll of the dice; seven yards of cotton; seven miles from here; seven acres of land; seven degrees of incline; seven degrees below zero; seven grams of gold; seven pounds per square inch; seven years old; finishing seventh; seven thousand dollars of debt; seven percent of alcohol; The Magnificent Seven. How can an idea with one name be used in so many different ways, denoting such various senses of quantity?"
(Kilpatrick)

Number sense can be described as someone's good intuition about numbers and their relationship (Howden, 1989). We could question the students to tell us the first thing that came to their mind when we said, "twenty four". When they gave answer like, "two dozen of donuts", "the whole day", and "the age of my aunt", instead of only made the drawing of two and four, it means that they have a special 'feeling' for number. They have an intuition about how the numbers related to each other and the world around them.

Why is number sense important? When during the process of learning mathematics students were only trained to master the algorithm and the basic facts, they would not custom to explore the relation between numbers and only mastered the ready-made mathematics. They would lose the meaning of mathematics itself and would see the mathematics as a set of formula that should be remembered by heart. They could not see the connection between mathematics they learn to their daily life. This is contradicted with Freudenthal idea that stressed mathematics as a human activity. According to Freudenthal, mathematics must be connected to reality, stay close to children and be relevant to society, in order to be of human value. Howden (1989) stated that number sense built on students' natural insights and convinced them that mathematics made sense, that it was not just collection of rules to be applied. Having the number sense, students can make judgement about the reasonableness of computational results and can see their relation with daily life situation.

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Many studies about the development of number sense have been conducted and some proposal about the frameworks and the activities about this have been suggested (For example: Griffin (2005), Jones et al. (1994), McIntosh et al. (1992), Howden (1989)). One of the more special studies to collaborate mathematics education and neuroscience is being conducted by Fenna van Ness and Titia Gebuis in Mathemathics and Neuro-Science Project (MENS Project). In that study, the process of the children achieve number sense was tried to be associated with their spatial thinking skill, which one of it is structure. Structures and structuring have been believed to be an important mathematical idea and activity toward the process of the growth of early number sense in a child. The structures and structuring would give help to a child on perceiving numbers. The preliminary results of that study showed improved mathematical achievement, suggesting that explicit instruction of mathematical pattern and structure can stimulate student's learning and understanding of mathematical concepts and procedures. Some children recognized the spatial structures that were presented and knew to implement these spatial structures for simplifying and speeding up counting procedures.

Inspired by that research and the vision of Freudenthal, we tried to develop a local instructional theory on guiding the development of number sense with the support of structuring for young learners on age 6 or 7 based on Indonesian's contextual situation. It means the activities of structuring and the structures that were used in the learning instruction were meaningful and stayed close to Indonesian children situation. The kind of structures that were used was structure that was adapted to Indonesian's context and situation, not only an adoption from the structure that was commonly used in the Netherlands. These structures should be kind of structures that are recognizable and meaningful by Indonesian students.

In developing the local instructional theory, we combined study of both the process of learning and the means that support the process. Thus, our research aimed to:

1. explain children's thinking process and achievement in exploring structure and structuring in the relation on how they perceive numbers;
2. support children's number sense growing process especially in counting by using their ability on structuring.

To explain children's thinking process and achievement, the research will be guided and will answer these following questions.

a. What is the role of different structures and structuring in the relation on how a child perceives numbers?
b. How can children at the early development use the structure to support their growing

process of number sense especially on counting?

c. What is the role of socio-mathematical practice in motivating an individual's number sense development?

And for the second aim the research would try to answer these following research questions.

a. What kind of contextual situation, means and instruction that support children's number sense growing process through structuring and symbolizing?
b. What kind of activities that stimulate the emergence of socio-mathematical practice that motivate an individual's number sense development?

We conducted this study using design research as the methodology. Design research is said as one way to develop an instruction theory and can yield an instruction that is both theory-driven and empirical based (van den Akker et al., 2006). By the design research the relevance between the research and the educational policy and practice could be maintained.

We present the result of this study in this thesis as below. After giving our purpose in this chapter, we will explain the theoretical framework of this study in chapter 2. Then, in chapter 3 we will describe the design research as our method in this study. We will also clarify about our data collection and describe our intended data analyses in chapter 3. In chapter 4, we will present the hypothetical learning trajectory (HLT) as the basis of this study. The result of the teaching experiment will be analyzed on chapter 5. Then we will discuss and make conclusion of this analyses on chapter 6. In this last chapter we will also proposed the refined HLT that can be used for the next cyclic of the study.

2. Theoretical Framework

Sense of number patterns is a key component of early mathematical knowledge.
(N.C. Jordan)

2.1. Motivation for the research

Most of the time, mathematics in Indonesia is taught in a very formal way and the process of learning is merely a transfer of knowledge from the teacher to the students without deep understanding. This fact set off an unstable foundation for mathematics in the higher level and created an idea that mathematics is only a set of formula that should be remembered by heart and it is not connected to the problems in the daily life.[3]

One interesting observation on the incident when a child on the 4th grade was asked to solve the sum of 886 – 807[4]. First he wrote the sum vertically.

Figure 1. Illustration of counting problem in Indonesia

Then he subtracted 6 with 7. Then he started using his fingers and toes to made 16 and 7. He took away 7 from the 16. To know the result, he counted one-by-one his fingers and toes. He seemed not convinced with the result. Then again he repeated using his fingers to express the numbers and count them one-by-one to get the result. Here we saw that he did the task in a very algorithmic way. And one thing that looked so surprising was he solved the problem up to 1000 using the first grade strategy, i.e. one by one counting.

It is an example of a false impression. In one side this child seemed be able to solve this problem. But in another side it seemed that this problem had no more meaning for him than just subtracting numbers. His strategy was not developed through his learning process. He might not have the flexibility to choose the more appropriate strategies on solving the problem. This example is not the only one which happened during the process of learning. This condition yielded anxiety for mathematics educators in Indonesia because inflexibility with number set off the weak point on learning and doing mathematics. To prevent this

[3] http://209.85.135.104/search?q=cache:VtueE1KHw1UJ:www.mes3.learning.aau.dk/Projects/Fauzan.pdf+mathematics+education+in+Indonesia&hl=nl&ct=clnk&cd=5&gl=nl . Consulted on 27 Oct. 08
[4] http://www.teachertube.com/view_video.php?viewkey=7e18e50b58eb7e0c6ccb . Consulted on 27 Oct. 08

problem, the math educators tried to find the way to help the students to be more familiar with numbers, have the meaning of numbers, and understanding the relation and the operation on numbers.

In this study we proposed to support children's development of counting ability as the part of the development of the number sense by structuring. Our proposal is inspired by The Mathematics Education and the Neuroscience (MENS) in the Netherlands. The MENS project was initiated to integrate research from mathematics education with research from educational neuroscience in order to come to a better understanding of how the early skills of young children can best be fostered for supporting the development of mathematical abilities in an educational setting. The focus of MENS project was on the development of awareness of quantities, on learning to give meaning to quantities and on being able to relate the different meanings of numbers to each other.

We developed a local instruction theory for developing number sense especially for young children on age 6 or 7 based on Indonesia contextual situation with the support of structuring. We used the realistic mathematics approach on creating the activities, so that the activities would be experientially real for the students.

2.2. Number Sense

Number sense is understood as someone's intuitive feel and flexibility on numbers and the relationship between numbers. People who have a good number sense can develop practical, flexible, and efficient strategies to handle numerical problems and make mathematical judgment (Howden, 1989; Greeno, 1991; McIntosh et al, 1992; Treffers, 1991). People can have a better number sense when they are accustomed to explore the numbers, visualize the number in various contexts, and could relate the numbers in ways that are not limited by traditional algorithms. Exploring the numbers and visualizing them in various contexts enable students to sharpen their feeling about numbers. They may see numbers in their daily situation and environment around them. For them who have a good number sense, a number will have more meaning than just a symbol or drawing. The "twelve" is not merely meant "12", but also "two packs of diet soda", "a half dozen", or "numbers on a clock". They may have representations for each number. These representations can be an image for the children to help them getting better understanding about the connection among numbers and the connection between number and the reality. When they are in a habit of this, they may easily see the relation among numbers. They may create these relations in their own way, not always as what the traditional algorithm said. They may see that 10 could be 5 and 5, or 4, 2 and 4, or 2, 2, 2, 2 and 2, and many more. In their upcoming learning experience, this sense will help them to be more flexible on choosing the strategy to solve an arithmetic problem. The more ideas about the numbers they have, the more flexible they choose the strategies.

Early number idea on a child's mind is begun with counting (van den Heuvel – Panhuizen, 2001; Griffin, 2004). Counting for an adult seems natural and simple, but for a child counting is a growing and accumulative process. In his didactical phenomenology, Freudenthal (1983) stated that many children count before having constituted number as a mental object. He distinguished:

1. counting as reciting the number sequence;
2. counting something, means connecting the numerals with the set that is counted or produced;
3. interpreting after counting the counting result as number of the counted or produce set.

Children have developed the number sense when they count with understanding and recognize how many in sets of objects. When a child is asked how many candies are in the table, and she answer, "one… two… three…" instead of "one… two… three… three!" shows that she has not had the idea of counting. Instead, she does the counting sequence as a verse to recite. The children also need to develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections. It includes the ability to compare numbers, order numbers correctly, and to recognize the density of numbers. In this necessity, children shall comprehend that a set of number is invariant over time and transformation but depend on the condition (Freudenthal, 1983). A set of five dots will always be the same set as yesterday, today or tomorrow. It also still the same set when we shift the dots in the different position. But it will be changed when something is added or taken away.

In general, perception of number is the ability to discriminate, represent, and remember numbers (Starkey P. & RG Copper, 1980). Thus, in this research we defined that children have perceived numbers when they become aware about quantities, be able to give meaning to quantities, and be able to relate the different meanings of numbers to each other.

2.3. Structure and Structuring as Part of Spatial Skill

For acquisition of the concept of numbers it is required the constitution of certain relational patterns between the numbers (Freudenthal, 1983). Papic and Mulligan (2005) defined a pattern as a numerical or spatial regularity and the relationship between the elements. In this regularity and the relationship, spatial structure – or in short we call structure

– is the important element. Structure is a product of act of organizing space – that was coined by Batista (1999) as spatial structuring. He defined:

> Spatial structuring is the mental operation of constructing an organization or form for an object or a set of object. Spatial structuring determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationship between component and the new object. (p.418)

Spatial structuring – that from now and later we called structure - is one fundamental method for children to organize the world (Freudenthal, 1987). Structure and structuring are part of the spatial skill. Tartre (1990) considered spatial skill to be those mental skills concerned with understanding, manipulating, reorganizing, or interpreting relationships visually. To interpret relationship visually, children needs the spatial visualization skill. Children's spatial visualization skills contribute to their ability to organize representations of objects into spatial structures (such as dice configurations and finger images). These spatial structures relate to the children's conceptions of shapes with which they become familiar through exploring their surrounding space (van Nes & Jan de Lange, 2007).

Children's concepts of quantities and number, then, may greatly be stimulated when children are made aware of the simplifying effects of structuring manipulative (van Nes & Jan de Lange, 2007).Some children may familiar with some specific structures, like dots on dice, finger counting images, rows of five and ten, bead patterns, and block constructions. Using these structures children may have the images of a number. When they asked for three, instead only the symbol 3, they may say that three is three dots on a die, or three is pointing finger, middle finger and ring finger. Structured object may not help the children for having the image of number when the children are not familiar with the structure or can not recognize the structure.

Structure may help the children not only on perceiving the number by having the images of the number. Structure may also help the children to develop the idea in mental arithmetic. For instance, string of beads in fives structure helps the students to have the strategy of skip counting, or the friendly number (of ten). And exploring symmetrical structures gives opportunity for the children to having the idea of double number (and almost double).

In accordance with van Nes (2007) we proposed that once children can imagine (i.e. spatially visualize) a spatial structure of a certain number of objects (i.e. configuration of objects that makes up a shape) that are to be manipulated, then learning to understand quantities as well as the process of counting (i.e. emerging number sense) should greatly be simplified.

2.4. Realistic Mathematics Education

Realistic Mathematics Education (RME) is a theory of mathematics education that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing instructional materials for mathematics education (Bakker, 2004). RME is rotted in Freudenthal's interpretation of mathematics as an activity that involves solving problems, looking for problems, and organizing a subject matter resulting from prior mathematizations or from reality (Gravemeijer, Cobb, Bowers, & Whitenack, 2000).

The instruction in RME should (a) be experientially real for the students, (b) guide students to reinvent mathematics using their common sense experience, and (c) provide opportunities for students to create self-develop models. Experientially real problem often involve everyday life setting or fictitious scenarios, but not necessarily so. For the more advance students, a growing part of mathematics itself will become experientially real. The word experientially real is related to the culture and tradition of subjects. The story of a gnome who stole dots on a mushroom to evoke counting strategy will be experientially real for the children in the Netherlands, but can be very artificial for the children in Indonesia as they never heard about gnome and never seen mushroom with dots. Common sense experience is also related with children's daily life experience. The symmetrical property on the butterfly wings to suggest the doubling idea is acceptable for the children in Indonesia since that is the condition of the butterfly as they see everyday. But for the children in the Netherlands, this symmetrical property of the butterfly wings may not easily acceptable as it is not their common sense. Using the common sense of the students, first developer tried to imagine the route of the class might invent. Then they tried to build on students' informal modelling activity to support the reinvention process. The instructional sequence should provide setting which students can model their informal mathematical activity. This students' *model of* informal mathematical activity evolve into *model for* increasingly sophisticated mathematical reasoning (Gravemeijer, 1999).

3. Methodology

Method helps intuition when it is not
transformed into dictatorship
(Mihai Nadin)

3.1. Design Research as the Research Method

On conducting this experiment we chose design research as a method. We chose this method as we see that design research (a) offers opportunities to learn unique lessons, (b) yields practical lessons that can be directly applied, and (c) involves researchers in the direct improvement of educational practice (Edelson, 2002). We expected that the result of our research that was conducted by this method could give real and direct contribution on the learning process in the classroom. As we used design research as the methodology, we followed the three phases of the design research which are (a) preparation and design phase, (b) teaching experiment phase, and (c) retrospective analysis phase.

During *the preparation and design phase*, first we chose number sense as our domain of research. We were in line with Howden (1989) who defined number sense as a person's intuitive understanding of numbers, the relations and operations between numbers; and the ability to handle daily-life situations that include numbers. We chose number sense as the domain since number sense plays an important role for the basic mathematics development, especially for young children. For the instrument during this research, we use the hypothetical learning trajectory (HLT). An HLT can make a link between an instruction theory and a concrete teaching experiment and offers description of the key aspects of planning mathematics lessons. This HLT was defined by Simon (1995) as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities.

Throughout this phase, we collected activities that has potential role in the HLT towards the end goal. We carried out this collection in three manners: studying the findings from the previous relevant researches especially the MENS project, gathering mathematical phenomenology concerning the domain, and conducting the task interviews and small try outs with the children in the age – range as the experimental subject. As the result in the end of this phase we developed a sequence of activities in a well – defined HLT. This description of the HLT will be described in chapter 4.

The teaching experiment phase was started in the beginning of the first semester of the first grade in the academic year 2008/2009. During the teaching experiment, the HLT functions as the guideline for the teacher and researcher what to focus on in teaching, interviewing, and observing. Sometimes the teacher or the researcher feels needed to adjust the HLT. Using the design research as the research methodology it is possible to redesign the instructional activities as the design research is a cumulative process of thought experiment and instruction experiment (figure 2).

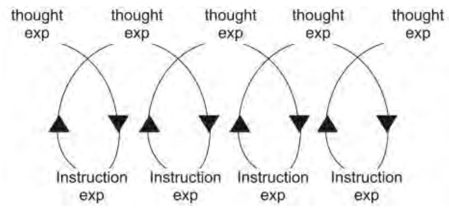

Figure 2. A cumulative cyclic process

We analyzed the daily lesson in a short retrospective analysis to control the consistency between the practices and the conjectures. This daily retrospective analysis might result the changes of HLT in the middle of teaching experiment. We used the result of the daily retrospective analysis to refine the HLT for the next activity.

The last phase we conducted was *the retrospective analysis* for the whole teaching experiment. During this phase, the HLT functioned as the guidelines for the researcher what to focus on in the analysis. After this retrospective analysis, the HLT could be formulated and yielded the refined HLT that can be used as the guide in the next research cyclic. We will describe the result of this retrospective analysis in chapter 5.

3.2. Data Collection and Data Analysis

We conducted our experiment in SD Bopkri III Demangan Baru Yogyakarta. Twenty students were involved in this experiment. This school has been involved in the PMRI[5] project since 2003, under the supervision of Sanata Dharma University[6].

Before we started the experiment, we interviewed the headmaster and the teacher of the first grade. We questioned them about the classroom culture of the grade one in that school. We tried to collect the data about the characteristic and the background of the

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[5] PMRI stands for Pendidikan Matematika Realistic Indonesia. The main mission of PMRI is to improve the quality of mathematics education in Indonesia by implementing an innovation approach in mathematics education called Realistic Mathematics Education (RME). http://www.pmri.or.id/en/index.php . Consulted on 27 Oct. 2008

[6] http://www.pmri.or.id/lptk/lptk2.php?id=6 . Consulted on 27 Oct. 2008

students, and also the norms and beliefs they had about the learning process. We also made the video observation to see the classroom condition.

After that, we interviewed the group of 2 to 10 students. We did these interviews to know their knowledge of numbers and their familiarity with structure. Our aim by this interview was getting the input about the didactical phenomenology of Indonesian students about number, structure and structuring.

Then, we tried to test our first design of HLT. But due to the time limitation we only test the important activities. We test these activities in a group of ten students. We used the result of this short experiment to look over our first HLT and yield the second HLT. This second HLT will be described in chapter 5.

The real teaching experiment was conducted on 15 July – 20 August 2008 with 20 first graders as the experimental subject. First, we conducted a pre-test. Since our experimental subject was not the group we worked with before, this pre-test was aimed to know the starting point of the students. We gave several tasks for the students to see their knowledge about numbers and the structure. After that we tested the second HLT. We recorded the learning process with the video. We also used the worksheet to record the students work. Last, we interviewed 10 students about the numbers, structures and structuring to see the evidences of their learning process.

The videos of interviews, tests and experiments were observed together with the paper works from the students. The researcher consulted the results with the supervisors to analyze the learning process. We chose some important moments which showed the development of children's number sense during the learning process. As we said before, during the analysis phase, the HLT functioned as the guidelines for us what to focus on in the analysis. Thus, on observing the incidents and the results of the students work, we referred to the HLT. The results of the analyses of the teaching experiment would be an explanation and ratification of the conjectures of children's thinking process but also can be used to refine the conjectures and HLT for the next cyclic.

4. Hypothetical Learning Trajectory

To teach math, you need to know three things.
You need to know where you are now.
You need to know where you want to go.
You need to know what the best way to get there.
(Sharon Griffin)

Our endpoints of this research are supporting children's perception about number and their counting ability as a part of the development of number sense. On giving this support, we elaborated structuring and structure, which included pattern and symmetrical pattern. We developed the activities that helped students on perceiving the numbers by having the images and the meaning of the numbers. At the same time these activities were aimed to develop their counting strategy. We evoked the idea of doubling, almost doubling, and skip counting as an alternative for one-by-one counting. Having done the activity, students may see the relation among numbers in the concrete and abstract level.

We developed the instruction in realistic approach. Thus, the activities in our instructions would be meaningful for the students and meet with their common sense. We began with observing the butterfly wings. When we see from the mathematic point of view, the butterfly wings have a very interesting fact. The both side of the wing of a butterfly are patterned and the pattern of the left side and the right side are symmetrical. From the Indonesian students' interest, butterfly is very attractive. They used to play in the garden with the butterflies or even caterpillars. They experienced and had seen that there is symmetrical property in the butterflies' wings. They had belief that butterfly wings are always symmetrical, although in their daily life they would use the term "the same" on mentioning this property instead of symmetry. Mathematically, these two terms are not the same. The structure of my right hand and the right hand of my friend are *the same*, but the structure of my right hand and my left hand are *symmetrical*. Nevertheless we accept that as the starting point of the children.

We found that this contextual situation was powerful as a mean to reach our endpoints. The pattern on the butterfly wings can motivate the students to structuring the objects to represent the numbers. And by having experience of structured object, the children would develop the image of the number and could relate numbers with other numbers and their world. The symmetrical property of the butterfly wings would encourage the students to see the double and also the almost double. Seeing the pattern on the one disk of the wings, children could predict the pattern on the other side since they are symmetrical.

Return to our theoretical framework, it was said that structure was not meaningful for a child when they did not recognize it. Thus we developed activities about the structure recognition that was presented as the first activity preceding counting strategies. Our design encompassed two portions: recognizing structure to develop the number image and building the relation among the numbers and employ structure, especially symmetrical structure, to develop the counting strategy, i.e. doubling, almost doubling and skip counting. In summary our skeleton of sequence for supporting children's number sense development would be explained as the following.

1. Developing Context and Recognizing Structure.
   Using their common sense about the symmetrical property in the butterfly wings, students start recognizing special structure in the butterfly wings. The students discussed the phenomena using the term "the same" or "similar" or "symmetrical". Then the students started structuring the dots in the symmetrical pattern to express the quantity of a set of dots on the butterfly wings.
2. Various Constructions of Structure.
   The students proposed various kind of structure to express the particular number. While doing so, the students develop a framework about various representations for a number and the relation for a number to each other.
3. Organizing the structure.
   Using a table, students would organize the various representation of a number. This type of activity would offer students the strategy to shorten the counting process by recognizing the relation among the number. Students would start to build the doubling and almost doubling idea.
4. Building the formal idea.
   The students develop the fundamental methods for arithmetical reasoning based on the framework of number relation.
5. Applying the structure to represent a number in formal level.
   The students would work in formal level to reason about the composition or decomposition of a number. Students would develop the strategy of skip counting.

As we have said before, design research should also forms conjectures about the potential mathematical argumentation and the cascade of tools and imageries (Gravemeijer, 2003). Hence we pictured that this hypothetical learning trajectory would work out in the following manner.

a. Activity 1:

The starting point for doing this activity is students' common sense about the symmetrical property on the butterfly wings. By this activity, we invited the students bring this common sense into awareness. The sequence started with investigating pattern in a butterfly wings. We envisaged that by investigating patterns on the butterflies' wings and looking at its characteristic, students would gain the idea about symmetrical in an understanding. The students would discuss the pattern on the butterflies' wings, compare those patterns, and find the resemblance among those patterns – that is the symmetrical property. The students might use various words on discussing the phenomena as the term "the same" or "similar" or "symmetrical". We expected that even though the students did not mention about symmetrical, they could recognize that the right side of the butterfly wings was symmetrical with the left side.

b. Activity 2:

Having the awareness of the symmetrical property on the butterfly wings from the previous activity as the starting point, in this activity, students started to make arrangement of dots on the butterfly wings. In term of mathematical goal, students tried to represent the number with the set of structured dots. We expected that the students can construct an amount that is being asked and reason using structure. On arranging the dots, students might do that in symmetrical pattern since they have already the imagery of learning from the previous activity.

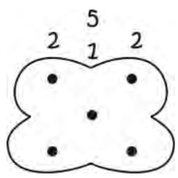
They discussed in paired about the possible arrangement of the dots to represent a number. The numbers that were demanded should start with the evens. An even number would enable the students to use their understanding of symmetrical pattern during arranging the dots. "Four" would be a good choice, since "four" is a small number but give more possibility to the given answer than "two". Then continue with 6, 8 and 10. It was predicted that the students will put the dot evenly in both side of the wing. But it was also possible that may happen a child does not put the dots evenly.

After discussing the even, the teacher could start to demand the odd numbers. The odd could be very difficult since to make it keep symmetrical, students needed to decide putting at least one dot in the middle. By putting the dot in the middle seems that student understood the idea of half to make it fair. Half part for the left side, half part for the right side. The students might not aware with this understanding, but they could use it to solve the problem.

Just like pattern and symmetrical, although in this activity the students used the idea of even, odd and half, they did not have to know the words.

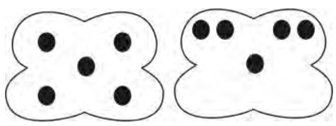
c. Activity 3:

The mathematics goals in this activity were: students could represent the number that was being asked and reason using structure. The starting point for this activity was children's experience on arranging the dots in the symmetrical pattern to represent a number. In this mini lesson activity, students needed to arrange the dots in the butterfly wing as much as the number demanded. The teacher guided them to put the number above the wing that represents the number of the dots in the left, in the middle, and in the centre, and also the total. They numbered the wings in the reason that they needed to know the kind of the butterfly wings they had. In the later lesson this number was very powerful to guide the students to see the relation between numbers. For example, one possible arrangement for five dots was



The "5" was the total of all dots, the "2" in the left was the number of dots on the left wing, the "1" in the middle was the number of dot on the middle of the wing, and the "2" in the right was the number of dots on the right wing.

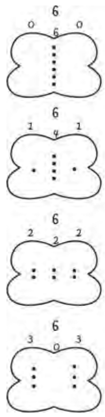
It was predicted that the students will not distinguish the formal level similar structure. For example:



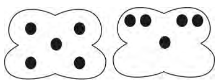
Since visually it looked different, students might recognize this structure as the different structure. Although when they had to write the number, they get the same arrangement: 2 – 1 – 2 (two on the left side, one on the middle, and another two on the right side).

d. Activity 4:

The mathematics goal for this activity was that students could see the relation among numbers. The starting point of this lesson was the students' knowledge about the existence of various representation of a number. A table was introduced to guide the students to see the relation among the numbers. Using this table, students were guided to move to the formal level. The students would collect all the possible structure for a number, and arrange them in one table.



	6	
0	6	0
1	4	1
2	2	2
3	0	3

The students discussed in pairs to record all possible arrangement of dots on the butterfly wings for a certain number. In this activity, students would be confronted with a dilemma to decide whether  were different arrangement or not. In this step they were challenged to come to more understanding about the structure and the number representation.

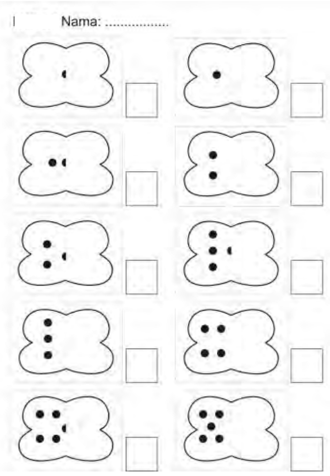
e. Activity 5:

The mathematics goal for this lesson was strengthen the students' idea about the doubling and almost doubling. In the later learning process, the doubling and almost doubling idea would be a powerful strategy to solve the arithmetic problems. The starting point for this activity was students' knowledge about the various representation of a number. The story about the butterfly which was perching on a leaf so that they could only see one side of the wing could be a lead discussion about the idea of double and almost double. The students discussed the method to determine the amount of the dots in the both wings. In this activity student might also discover the strategy of counting on. They knew already the amount of the dots on the one side. Thus to know the total they recounted the visible dots.

There were three conjecture of students' thinking in this activity. The first one was when they struggling finding the answer by counting the dots two time. So, to tell the number of the dot on those butterfly wings they have to sum the number of the dots in the two sides. The second one was when the children capture the number of the dots on the butterfly wing at once and counting on the dots to get the total. In these two incidents the students may do this without realizing that they do the addition. The last one is when the students still do not have the idea about symmetric or doubling. These children may only write the number of the dots that they see.

f. Activity 6:

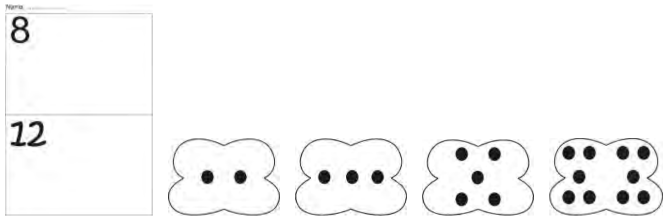
With the same goal and the same starting point as the previous activity, the lesson then continued with indivual work. This repetition of the goal was aimed to emphasize the double and almost double idea. The students were asked to complete the wing by making a drawing of dots on the other side of the wing.



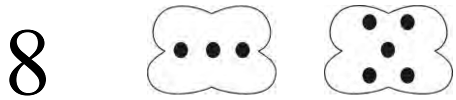
There were three conjectures of students thinking in this task. The first one if the students really got the idea of symmetrical or doubling. Then they would not have any difficulties to complete this task. The second one when they already got the idea of symmetrical or doubling but they do not know how to handle the half dots, they would not accomplish this task successfully. Some children might see this half as one complete dot and do not realize that it was a half, thus they mirrored this half dot and they get one extra dot. Some children might see this half dot also as one complete dot but they do not mirror this half dot since it stay in the middle of the butterfly wings. The third was when the children do not have any idea about the task.

g. Activity 7:

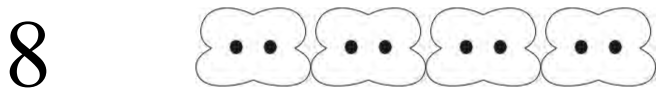
In this activity, the students' interest moved from working with a single butterfly to seeing butterfly as a group. Students were asked to make a group of butterfly wings which the total amount of the dots on their wings fulfilled a demanded number. The mathematics goals in this activity were that students could relate, compose, and decompose numbers. There were four kind of dots arrangement in the dotted butterfly: dotted butterfly with two dots on it, with three dots on it, five dots on it, and ten dots on it.



There were two conjectures of students thinking on fulfilling this task. The first one was when the students think that they could use any number of the dotted. For example, to fulfil the 8, they stuck butterfly with 3 dots and 5 dots.



The second one was when the students decide to use patterned number to fulfil the number. For example, to fulfil the 8, they stick four times 2 dots butterfly.

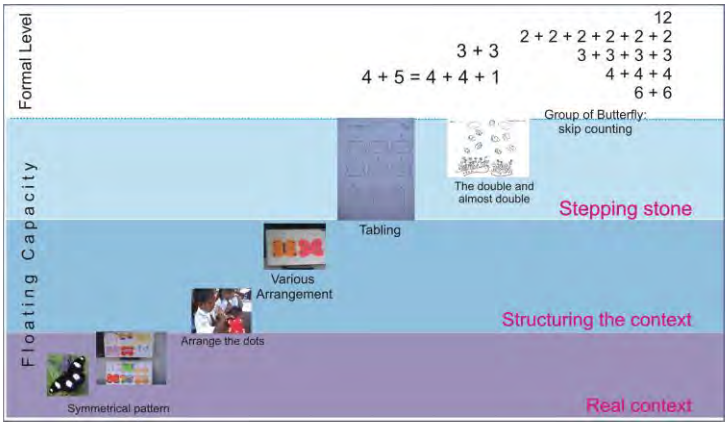


There might two different strategies that students use in this activity. The first one was when the students tend to use trial and error strategy. They picked up the wings, count the dots then glue the wing. After that they counted the dots again. When it was okay then they moved to the next task, but when it was not okay then they unglued the wing and changed with other wing. They might do this several time until they get the answer. The second one was when the students well plan the wing thus they did not need to pick and change many times. Usually it happened to the students who chose the patterned number.

h. Activity 8:

This activity had the same mathematics goal as the previous activity. The starting point for this activity was students' experience in the previous activity. The task was still the same, that was making a group of butterfly wings which the total amount of the dots on their wings fulfilled a demanded number. The difference was in this activity students were stimulated to move to the formal level. The dots in the butterfly wings were replaced by the number represented the amount of the dots. The shift from working with dotted wings to working with numbered wings was related to the model of – model for transition. Initially, the students' work with the model would foster the framework about the number relation. After the transition the model would became a model for generalized mathematical reasoning.

In a nut shell, the description of the HLT above can be pictured in the schema of the learning line below.



In that schema, we can see that the activities are tried to be built in realistic mathematics approach. The progressive mathematization appears, since the activities are built in the horizontal and the vertical direction. The activity is started in the real context level and directed to the formal level. In this research, the formal level has not been investigated yet due to the limited time. And it is considered that the formal level could not be achieved in a very short lesson.

The Cascade of Tools and Imagery

The first tool that was used was picture of butterfly wings. This tool was used to give meaningful insight for the students and invite students became aware about their common sense of symmetrical property on the butterfly wings. The second tool, butterfly wings and the dots model was used to keep the insight in students mind when they were doing the activity. The next tool was a table that was introduce by the teacher. This table would help the students to see the relation among numbers. This relation in the later activity would be develop as the idea of double and almost double. The group of the butterfly models with dotted would give help for the students to develop the skip counting strategy.

This succession of tools assured us about the comprehensiveness of the activity. The cascade of imagery were hoped to be a useful way to describe how the proposed sequence of tools can be seen as reflecting RME's theoretical reinvention process.

The table bellow shows the summary of the role of tools that we proposed.

Tool	Imagery	Activity/ Taken-as-shared interest	Potential mathematical discourse topics
The pictures of butterflies' wings		Recognizing the structure	Structure, symmetrical
Butterfly wings and the dots models	Symmetrical idea	Constructing structure	Even and odd number, half, structure, number image
Table	Various construction of dots	Structuring the structure by tabling all possible construction	Relation among numbers.
Group of butterfly	Double number	Composing and decomposing numbers	Double, almost double, compose and decompose number, skip counting strategy

1. Introduction

"Seven. What is seven? Seven children; seven ideas; seven times in a row; seventh grade; a lucky roll of the dice; seven yards of cotton; seven miles from here; seven acres of land; seven degrees of incline; seven degrees below zero; seven grams of gold; seven pounds per square inch; seven years old; finishing seventh; seven thousand dollars of debt; seven percent of alcohol; The Magnificent Seven. How can an idea with one name be used in so many different ways, denoting such various senses of quantity?"
(Kilpatrick)

Number sense can be described as someone's good intuition about numbers and their relationship [Howden, 1989]. We could question the students to tell us the first thing that came to their mind when we said, "twenty four". When they gave answer like, "two dozen of donuts", "the whole day", and "the age of my aunt", instead of only made the drawing of two and four, it means that they have a special 'feeling' for number. They have an intuition about how the numbers related to each other and the world around them.

Why is number sense important? When during the process of learning mathematics students were only trained to master the algorithm and the basic facts, they would not custom to explore the relation between numbers and only mastered the ready-made mathematics. They would lose the meaning of mathematics itself and would see the mathematics as a set of formula that should be remembered by heart. This caused them to be unable to see the connection between mathematics they learned with their daily life. This is in contrast to Freudenthal idea that mathematics will be a human activity. According to Freudenthal, mathematics must be connected to reality, stay close to children and be relevant to society, in order to be a human value. Howden [1989] stated that number sense built on students' natural insights and convinced them that mathematics made sense, that it was not just collection of rules to be applied. Having number sense, students can make judgement about the reasonableness of computational results and can see their relation with daily life situation.

Many studies about the development of number sense have been conducted and some proposal about the frameworks and the activities about this have been suggested (For example: Griffin [2005], Jones et al. [1994], McIntosh et al. [1992], Howden [1989]). One of the more special studies to collaborate mathematics education and other scientific discipline is being conducted by Fenna van Ness and Titia Gebuis in Mathemathics and Neuro-Science Project (MENS Project). In that study, the process of the children achieve number sense was tried to be associated with their spatial thinking skill. The preliminary results of that study showed improved mathematical achievement, suggesting that explicit instruction of mathematical pattern and structure can stimulate student's learning and understanding of mathematical concepts and procedures. Some children recognized the spatial structures that were presented and knew to implement these spatial structures for simplifying and speeding up counting procedures.

In my observation as a math educator, most of the time, mathematics in Indonesia is taught in a very formal way and the process of learning is merely a transfer of knowledge from the teacher to the students without deep understanding. This fact set off an unstable foundation for mathematics in the higher level and created an idea that mathematics is only a set of formula that should be remembered by heart and it is not connected to the problems in the daily life.

Inspired by MENS research and the vision of Freudenthal, throughout this research, I will try to find the way to support the students to be more familiar with numbers, have the meaning of numbers, and understanding the relation and the operation on numbers. My support will focus on awareness of quantities, on learning to give meaning to quantities and on being able to relate the different meanings of numbers to each other. To give the support, I will involve the pattern and structure as both of them can stimulate student's learning and understanding of mathematical concepts and procedures [Mulligan, et al., xxx; van Ness, 2007].

In accordance with Cobb [2003], in developing the local instructional theory, I combined study of both the processes of learning and the means to support the process. Thus, my local instructional theory aims to:

1. explain children's thinking process and achievement on perceiving numbers by patterning;
2. support children's counting strategy by patterning.

To accomplish those aims, in the end of this research I will try to answer these following research questions.

a. What are the roles of patterns to support children perceiving numbers?
b. How can children develop their counting strategies by patterning?
c. What is the role of socio-mathematical norms in motivating children on perceiving number and developing counting strategies by patterning?
d. What kind of contextual situation, means and instruction that support children to perceive numbers and develop counting strategy by patterning?

We conducted this study using design research as the methodology. Design research is said as one way to develop an instruction theory and can yield an instruction that is both theory-driven and empirical based (van den Akker et al., 2006). By the design research I expect that the relevance between the research and the educational practice could be maintained.

I present the result of the study in this thesis in this following manner. After giving my purposes in this chapter, I will explain the theoretical framework that underlies this study in chapter 2. Then, in chapter 3 I will describe the design research as our method in this study. I will also clarify about my data collection and describe my intended data analyses in chapter 3. In chapter 4, I will present the hypothetical learning trajectory (HLT) as the basis of this study. The result of the teaching experiment will be analyzed on chapter 5. In this chapter, in the conclusion section, I will answer my research questions mentioned above. In the last chapter, chapter 6, I will present my reflection about this research and also my own learning process reflection during doing this research. In the last sections I will proposed some recommendation to for the next cyclic of the development.

2. Theoretical Framework

Sense of number patterns is a key component of early mathematical knowledge.
(N.C. Jordan)

In this chapter, I begin the first section with the description about the number sense. Then I will continue with my explanation about structure and pattern, and how both of them take a part on the development of number sense. The following, I will continue with the description of realistic mathematics education (RME) which becomes my interpretative framework on designing the activity. Since in my design I also want to see how socio-mathematical norms give influence on individual's learning, in the last section of this chapter I will explain about the emergent perspective that includes socio-mathematical norms and mathematical practices.

2.1. Number Sense

Number sense is understood as someone's intuitive feeling and flexibility on numbers and the relationship between numbers. People who have a good number sense can develop practical, flexible, and efficient strategies to handle all kind of numerical problems [Howden, 1989; Greeno, 1991; McIntosh et al, 1992; Treffers, 1991]. People can have a better number sense when they are accustomed to explore the numbers, visualize the number in various contexts, and could relate the numbers in ways that are not limited by traditional algorithms. Exploring the numbers and visualizing them in various contexts enable students to sharpen their feeling about numbers. They may see numbers in their daily situation and environment around them. For them who have a good number sense, a number will have more meaning than just a symbol or drawing. The "twelve" is not merely meant "12", but also "two packs of diet soda", "a half dozen", or "numbers on a clock". They may have representations for each number. These representations can be an image for the children to help them getting better understanding about the connection among numbers and the connection between number and the reality. When they are in a habit of this, they may easily see the relation among numbers. They may create these relations in their own way, not always as what the traditional algorithm said. They may see that 10 could be "5 and 5", or "4, 2 and 4", or "2, 2, 2, 2 and 2", and many more. In their upcoming learning experience, this sense will help them to be more flexible on choosing the strategy to solve an arithmetic problem. The more ideas about the numbers they have, the more flexible they choose the strategies.

Early number idea on a child’s mind is begun with *counting* [van den Heuvel – Panhuizen, 2001; Griffin, 2004]. Counting for an adult seems natural and simple, but for a child counting is a growing and accumulative process. In his didactical phenomenology, Freudenthal [1983] stated that many children count before having constituted number as a mental object. He distinguished:

> counting, that is, reciting the number sequence;
> counting something, that is, in the counting process connecting the numerals with the set that is counted out or produced;
> interpreting after counting the counting result as number of the counted or produced set. [p.97]

Children have developed the number sense when they count with understanding and recognize how many in sets of objects. When a child is asked how many candies are in the table, and she answer, “one… two… three…” instead of “one… two… three… three!” shows that she has not had the idea of counting. Instead, she does the counting sequence as a verse to recite. As children progress in their ability to count, they discover easier ways of operating with numbers and they come to understand that number can have different representation [van Nes & de Lange, 2007]

The children also need to develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections. It includes the ability to compare numbers, order numbers correctly, and to recognize the density of numbers. In this necessity, children shall comprehend that a set of number is invariant over time and transformation but depend on the condition [Freudenthal, 1983]. A set of five dots will always be the same set as yesterday, today or tomorrow. It also still the same set when we shift the dots in the different position. But it will be changed when something is added or taken away.

In general, *perception of number* is the ability to discriminate, represent, and remember numbers [Starkey P. & RG Copper, 1980]. Thus, in this research we defined that children have perceived numbers when they became aware about quantities, be able to give meaning to quantities, and be able to relate the different meanings of numbers to each other.

2.2. Pattern and Structure to support the development of number sense

Before I go further to my exposition about the role of pattern and structure to support the development, first I will talk about the spatial sense – as both pattern and structure related to spatial sense. Freudenthal [1989, NCTM] defined spatial sense as the ability to ‘grasp the external word’. And Tartre [1990] considered *spatial skill* to be those mental skills concerned with understanding, manipulating, reorganizing, or interpreting relationships visually. To interpret relationship visually, children needs the spatial visualization skill. I am of the same opinion with Tatre [1990] that defined spatial visualization as the ability to manipulate object visually. Children's spatial visualization skills contribute to their ability to organize representations of objects into spatial structures (such as dice configurations and finger images). These spatial structures relate to the children's conceptions of shapes with which they become familiar through exploring their surrounding space [van Nes & de Lange, 2007]. To understand the term spatial structure, I draw on the definition of Battista [1999] to describe spatial structuring. He defined:

> *Spatial structuring* is the mental operation of constructing an organization or form for an object or a set of object. Spatial structuring determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationship between component and the new object. [p.418]

Then, I coin spatial *structure* as a product of organizing objects. Such a structure is an important element of a pattern [van Nes & de Lange, 2007].

Papic and Mulligan [2005] defined a *pattern* as a numerical or spatial regularity and the relationship between the elements – thus is its structure. I am in line with van Nes & de Lange [2007] that refered to structure as a configuration of object that relates to the component 'spatial regularity' in the given definition of pattern. But, in contrast with them, in this thesis I also look at the component 'numerical regularity' that refers to numerical sequence.

Papic and Mulligan [2005] also stated that pattern and structure are thus at the heart of school mathematics. According to them, *patterning* is an essential skill in learning, particularly in the development of spatial awareness, sequencing and ordering, comparison and classification. Freudenthal [1983] also saw that for acquisition of the concept of numbers it is required the constitution of certain relational patterns between the numbers. Children's concepts of quantities and number, then, may greatly be stimulated when children are made aware of the simplifying effects of structuring manipulative [van Nes & Jan de Lange, 2007] – as the activity of making symmetrical[1] configuration of dots on a butterfly wings model can stimulate their number perceptions.

[1] A figure is symmetric with respect to a line if it can be folded on that line so that every point on one side coincides exactly with a point on the other side. http://www.wtvl.net/honda/glossarypre.htm . Consulted on November 5th, 2008.

In accordance with van Nes [2007] I propose that once children can imagine (i.e. spatially visualize) a structure or pattern of a certain number of objects (i.e. configuration of objects that makes up a shape) that are to be manipulated, then their learning process to understand quantities as well as the process of counting (i.e. emerging number sense) should greatly be simplified.

2.3. Realistic Mathematics Education

Realistic Mathematics Education (RME) is a theory of mathematics education that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing instructional materials for mathematics education [Bakker, 2004]. RME is rotted in Freudenthal's interpretation of mathematics as an activity that involves solving problems, looking for problems, and organizing a subject matter resulting from prior mathematizations or from reality [Gravemeijer, Cobb, Bowers, & Whitenack, 2000].

The instruction in RME should (a) be experientially real for the students, (b) guide students to reinvent mathematics using their common sense experience, and (c) provide opportunities for students to create self-develop models. Experientially real problem often involve everyday life setting or fictitious scenarios, but not necessarily so. For the more advance students, a growing part of mathematics itself will become experientially real. The idea of experientially real is related to the culture and tradition of subjects. Freudenthal [1991] prefer to apply the term reality to what common sense experiences as real at a certain stage. The story of a gnome who stole dots on a mushroom to evoke counting strategy will be experientially real for the children in the Netherlands, but can be very artificial for the children in Indonesia as they never heard about gnome and never seen mushroom with dots. Common sense experience is also related with children's daily life experience. The symmetrical property on the butterfly wings to suggest the doubling idea is acceptable for the children in Indonesia since that is the condition of the butterfly as they see everyday. But for the children in the Netherlands, this symmetrical property of the butterfly wings may not easily acceptable as it is not their common sense. Freudenthal [*ibid.*] goes on to say that reality and what person perceives as common sense is not static but grows, and is affected by individual's learning process. Freudenthal [1971] proposes that the activity on one level is subjected to analysis on the next level, and the operational matter becomes subject matter on the next level. Thus the goal of realistic mathematics education then is to support students to creating new mathematical realities [Cobb & Gravemeijer, 2006] and this process (of guiding to create new mathematical reality – or I use now the word of Freudenthal: *mathematizing*) is viewed by Treffers [1987] in two manners: horizontal and vertical. In the horizontal mathematizing the students are guided to mathematize subject matter from reality, whilst in the vertical mathematizing the students mathematize their own mathematical activity. Instead of using the ready – made models, in this mathematizing process, RME looks for models that may emerge as *model of* situated activity, and then gradually evolve into entities of their own to function as *model for* more sophisticated mathematical reasoning [Gravemeijer, 1999].

2.4. Socio-mathematical norms and mathematical practices

As I have described in chapter one, I want to identify the role of socio-mathematical norms in motivating children on perceiving number and developing counting strategies by patterning. On answering that question, in this section I will discuss about the literature review of social aspect of learning. I will start this discussion with social norms.

Social norms refer to expected ways of acting and explaining that become establish through a process of mutual negotiation between the teacher and students [Cobb & Gravemeijer, 2006]. The social norms will differ significantly between classrooms that pursue traditional mathematics and those engage in reform mathematics. Since 2001, there is a reformation of mathematics education in Indonesia by implementing an innovation approach in mathematics education called Realistic Mathematics Education (RME). The paradigm of learning process is shifting from teaching paradigm into constructing students' own knowledge. The roles of the teacher which before were to explain and evaluate are now moving to guide and facilitate. Whilst, if in the old paradigm students had to figure out what teacher had in mind, now the norms move to explain and justify solutions, attempt to make sense of explanations given by others, indicate agreement and disagreement, and question alternative situations where a conflict in interpretations or solutions is apparent [*ibid.*, p. 31]. In my observation, this reformation practice is not an easy and a simple process. Teacher shall change their paradigms and the students need to learn the new norms.

Socio-mathematical norms can be distinguished from social norms as ways of explicating and acting in whole class discussion that are specific to mathematics [*ibid.*]. Cobb [1998] clarifies that the socio-mathematical norms include "… *what count as different mathematical solution, as sophisticated mathematical solution, an efficient mathematical solution and an acceptable mathematical solution…*[p.38]". Students develop personal ways of judging whether a solution is efficient or different and the teacher can not simply state what solutions are acceptable. For example, on determining whether dots configurations on the butterfly wings are acceptable or not, teacher shall not tell that the configurations are acceptable when they are in symmetrical pattern. Students shall reason in their own way to determine the accepted configuration.

The last social aspect of learning that I want to discuss is *mathematical practices*. Cobb and Gravemeijer [2006] describe mathematical practices as the normative way of acting, communicating and symbolizing mathematically at a given moment in time, which are specific to particular mathematical ideas or concepts. An indication that a certain mathematical practice has been established is that students' mathematical interpretations and actions constitute particular mathematical idea. When to represent "20" a student stuck four butterfly wings with five dots on each of them thus mathematical practice of skip counting strategy has became appeared.

3. Methodology

Method helps intuition when it is not
transformed into dictatorship
(Mihai Nadin)

3.1. Design Research as the Research Method

On conducting this experiment I chose design research as a method. As I have mentioned in chapter 1, I use the term of design research in accordance with van den Akker [2006] that says design research is one way to develop an instruction theory and can yield an instruction that is both theory-driven and empirical based. Cobb [2003] noted that the purpose of a design experiment is not to implement an instructional sequence and see whether it works; rather, the purpose is to use ongoing and retrospective analyses of classroom event as fodder for improvements to original design. I chose this method as I see that design research (a) offers opportunities to learn unique lessons, (b) yields practical lessons that can be directly applied, and (c) involves researchers in the direct improvement of educational practice [Edelson, 2002]. I expect that the result of my research that was conducted by this method could give real and direct contribution on the learning process in the classroom and at the same time can give contribution on instructional theory of number sense. As I used design research from a learning design perspective as the methodology, I followed the three phases of the design research which are (a) preparation and design phase, (b) teaching experiment phase, and (c) retrospective analysis phase [Gravemeijer & Cobb, 2006].

During *the preparation and design phase,* first I chose number sense as our domain of research. I am in line with Howden (1989) who defines number sense as a person's intuitive understanding of numbers, the relations and operations between numbers; and the ability to handle daily-life situations that include numbers. I chose number sense as the domain since number sense plays an important role for the basic mathematics development, especially for young children. Then I clarified the research goals in this domain. As I have presented in chapter 1, my goals are to explain children's thinking process and achievement on perceiving numbers by patterning and to support children's counting strategy by patterning. For the instrument during this research, I use the hypothetical learning trajectory (HLT). An HLT can make a link between an instruction theory and a concrete teaching experiment and offers description of the key aspects of planning mathematics lessons. This HLT was defined by Simon (1995) as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities.

Throughout this phase, I collected activities that has potential role in the HLT towards the end goal. I carried out this collection in three manners: studying the findings from the previous relevant researches especially the MENS project, gathering mathematical phenomenology concerning the domain, and conducting the task interviews and small try outs with the children in the age – range as the experimental subject. As the result in the end of this phase I developed a sequence of activities in a well – defined HLT. This description of the HLT will be described in chapter 4. This preparation phase had been done during February – April 2008 in the Netherlands.

The teaching experiment phase was conducted in the end of the second semester of the first grade in the academic year 2007/2008 and beginning of the first semester of the first grade in the academic year 2008/2009. In the first teaching experiment, I, my local supervisor and an observer tested some activities from the HLT that I had designed on April to a group of 2 – 10 students. From this first teaching experiment, we found that this HLT was not appropriate to support students' learning process. We found that some activities were too high in mathematics level for the students, were not realistic to the students and did not match with the classroom norms practice. Thus we – I, the teacher, the local supervisor and the observer – created the new HLT that we think more supportive to students' learning process. Again, we test some activities of this new HLT and found some ideas to refine this HLT – then we have another new HLT. This newest HLT was experimented on the 20 new first graders in the beginning of academic year 2008/2009 (July – August). During the teaching experiment, the HLT functions as the guideline for the teacher and researcher what to focus on in teaching, interviewing, and observing. Sometimes the teacher or the researcher feels needed to adjust the HLT. Using the design research as the research methodology it is possible to redesign the instructional activities as the design research is a cumulative process of thought experiment and instruction experiment (figure 2).

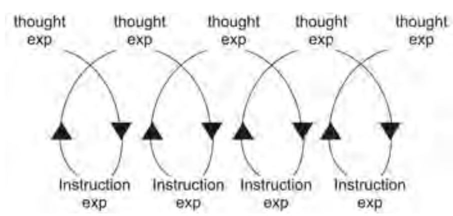


Figure 2. A cumulative cyclic process

I analyzed the daily lesson in a short retrospective analysis to control the consistency between the practices and the conjectures. This daily retrospective analysis might result the changes of HLT in the middle of teaching experiment. We used the result of the daily retrospective analysis to refine the HLT for the next activity.

The last phase we conducted was *the retrospective analysis* for the whole teaching experiment. During this phase, the HLT functioned as the guidelines for the researcher what to focus on in the analysis. After this retrospective analysis, the HLT could be formulated and yielded the refined HLT that can be used as the guide in the next research cyclic. We will describe the result of this retrospective analysis in chapter 5.

3.2. Data Collection

We conducted our experiment in SD Bopkri III Demangan Baru Yogyakarta. Twenty students were involved in this experiment. This school has been involved in the PMRI[2] project since 2003, under the supervision of Sanata Dharma University[3]. As our teaching experiment would be conducted in the new first graders, then we anticipated the students' ability on writing. We preferred to choose task activities, where the students show actions to perform the instruction rather than writing assessments.

Before we started the experiment, we interviewed the headmaster and the teacher of the first grade. We questioned them about the classroom culture of the grade one in that school. We tried to collect the data about the characteristic and the background of the students, and also the norms and beliefs they had about the learning process. We also made the video observation to see the classroom condition.

Then, we tried to test our first design of HLT. But due to the time limitation we only test the important activities. We test these activities in a group of two to ten students. We used the result of this short experiment to look over our first HLT and yield the second HLT. We also tested this second HLT and result some refinements for this second HLT and yield the third HLT that we used in the classroom teaching experiments. This second and third HLT will be described in chapter 5.

The classroom teaching experiment was conducted on 15 July – 20 August 2008 with 20 first graders as the experimental subject. First, we conducted a pre-test. Since our

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[2] PMRI stands for Pendidikan Matematika Realistic Indonesia. The main mission of PMRI is to improve the quality of mathematics education in Indonesia by implementing an innovation approach in mathematics education called Realistic Mathematics Education (RME). http://www.pmri.or.id/en/index.php . Consulted on 27 Oct. 2008

[3] http://www.pmri.or.id/lptk/lptk2.php?id=6 . Consulted on 27 Oct. 2008

experimental subject was not the group we worked with before, this pre-test was aimed to know the starting point of the students. We gave several tasks for the students to see their knowledge about numbers, pattern and structure.

After that, we experimented our third HLT in the classroom teaching experiment. During the classroom session, we recorded the learning process with the video. We tried to capture important moments that showed the emergence of mathematical practices. We also used the worksheet to record the students work. We used this worksheet to cross check our observation in the video. Finishing daily classroom session, I discuss with the teacher and the observer about the interesting happenings in the classroom and how we would conduct the activity in the following day. We tried to see the actuality of our HLT with students actual learning processes.

Last, we interviewed 10 students about the numbers, pattern, and structure to see the evidences of their learning process. We also recorded these interviews with the video.
I put the overview of all collected data in the appendix.

3.3.Intended Data Analysis

From the recorded learning lesson, I only chose some episodes that showed the emergence of mathematical practices and evidences of students' learning process. To keep the reliability of data interpretation, I discussed those critical moments from the video recording with my local supervisor in Indonesia. And after back to the Netherlands, to keep the reliability, I discussed my data interpretation with some of my fellow students and my supervisor. I also tried to keep the validity of my data by cross checking the findings from the correlated episodes on the videos, paper works from the students, and notes from the observer. In sample episodes presented in chapter 5, I use the word *teacher* collectively to refer to teacher, researcher, observer and local supervisor.

As I said before, during the analysis phase, the HLT functioned as the guidelines for us what to focus on in the analysis. Thus, on analyzing the incidents and the results of the students work, I referred to three components of HLT: (1) *The learning goals*: I analyzed my data collection – the video and the students' work – to see whether the students achieved the learning goal; (2) *The learning activities:* I found episodes in the actual learning series that became the evidence of the emergence of mathematical practice and the emergence of socio-mathematical norms; and (3) *The conjecture learning process*: I saw whether our conjectures of learning process happened in the actual learning process. From the result of the findings in the retrospective analysis, I would try to make a conclusion by answering my research questions. The last, these findings were also used make the reflection of the activity and some recommendations for the next cyclic development.

4. Hypothetical Learning Trajectory

To teach math, you need to know three things.
You need to know where you are now.
You need to know where you want to go.
You need to know what the best way to get there.
(Sharon Griffin)

On April 2008, before leaving the Netherlands to do the research in Indonesia, I have designed a Hypothetical Learning Trajectory. This HLT was tried out on May and June 2008 in Indonesia. From results of the try out and the discussion with the teacher and the local supervisor, I found that this HLT was not appropriate to support students learning process. Thus this HLT would be replaced by a new HLT that we thought more suitable for the learning process. The result of the try out, the discussion about the result and the teacher's and my local supervisor's suggestion, and the new HLT are presented in chapter 5.

I started designing this HLT with clarification of the goals as the end point. As we have mentioned in chapter 1, I aimed to (1) support children's achievement on perceiving numbers by patterning, and (2) support children's counting strategy by patterning. For those purposes I develop a sequence of activities that consisted of starting points of the children, the goal of the activities, the intended activity itself, and the conjectures of students' thinking.

In the starting point, I described the students' pre-knowledge or common sense about the contextual situation of the activity that made the activity became meaningful for them. In the goal of the activity, I described the mathematical goal that students would achieve during doing the activity. In the intended activity, I proposed activities that support them to reach the goals. I also showed what students can mathematically learn in these activities. Then in the conjectures of students' thinking, I explained what students might react when they did the activity. I also predicted about the students strategy and struggles on doing the activities.

Activity 1: Stand in the line

Starting point. We started this activity from their real experience that they always stand in the line before they enter the classroom. We would bring this real experience as the contextual situation of the activity that would be mathematized during the lessons. In their cognitive knowledge, we expect that students have already had competence on counting (as reciting the number sequence).

The goal of activity. After doing this activity, we expect that students would able to use the pattern of twos as the early knowledge to develop the skip counting strategy.

Intended activity. The teacher would start the sequence by asking four students to stand in a line just like before they enter the classroom. They the teacher asked two more students to joining the lines in front of the classroom. As the students doing the activity, they experienced on making the pattern of two: "2, 2, 2, 2, …". That 2 and 2 is 4, and add another 2 is 6, and add another two is 8, and so on. The students would predict the next number after 2 more students were added in the lines. After making the representation of this situation with drawing, the students get more image of pattern f two and how they constitute a number series.

Conjectures of the learning process. The hypothetical learning process for this sequence would be held like this. Before entering the classroom the teacher talks to the children that she was glad with their arrangement of standing in the line because she could easily see how many students presented that day. The teacher asked them to remember who was standing in their right or their left. Right after the class start, the teacher reminded the children about their arrangement of standing in line. Next, the teacher asked four children to stand up in the line, and asked the class how many children who were standing. The teacher asked the other four children who stood in the right or in the left children who were now standing and questions the children about how many children who are standing now and how they count. Ask more two children to stand up. Before joining them in the line, the teacher questioned them to predict the kind of arrangement that would possible, and where they have to stand. The teacher asked them to tell how many altogether and their reasoning. The discussion continued by advancing the problem, like: If there were 5 students stand only in one line, how many more students should join them to make a good arrangement of two lines? How many students would be altogether? For all the students in one class, how many children should stand in one side of the line, and how many in the other side. Does everyone have partner? How would the lines look like if there are two children sick, three children? The students could reason using the manipulative objects or their own drawing. On doing the reasoning we conjectured that not all the students used the pattern of two, instead they do one by one counting.

Activity 2: Packing of twos

Starting point. From the previous activity, students had already had the imagery of their learning: they could use and make the pattern of twos. This imagery would be occupied as the starting point for this activity. We also used their real experience about the packaging of juice, coca cola, snacks (that were arranged in pattern of twos). We would bring this real experience as the contextual situation of the activity that would be mathematized during the lessons.

The goal of activity. By doing this activity, students would have more mental image about the pattern of twos in their daily life. It is also expected that they could related numbers to each others by pattern of twos.

Intended activity. Students explored some kinds of packages that structured in pattern of twos. By doing this activity they would have more models to represent the configuration of a number in pattern of twos. They made and reason some representation by manipulative objects and drawing to show a number of objects using the pattern of twos. In this phase they develop their model of the situation.

Conjectures of the learning process. The teacher will bring various kinds of packing that occupy the pattern of twos. Then the students were asked to tell about what they observed in those packing. We expected that students would reason using their previous experience about the pattern of twos. Then the students would do the packing of some items (coca-cola, juice, snacks, candies, etc). We expected that the students would pack the items in pattern of twos as what they saw in their daily life. The next, students were asked to make a representation (by manipulative objects or drawing) to represent their packing and reported their result in front of the class. We expected them to use the numerical pattern of twos, even though we realized that on reporting the number of items they represented, the students tagged each item and connected it with each number word sequentially.

Activity 3: Exploring pattern and structures

Starting point. We used many kind of structured object from their daily life environment as the real experience of the students to be mathematized during the lesson. The imagery about the previous activity on patterning would be also the starting point of their cognitive knowledge.

The goal of activity. By doing this activity, students would be aware of structure and could use the structure to ease them on counting process. By occupying the structure, they did not need to count on by one – thus they counting process could be shortened.

Intended activity. In this activity children would be guided to start recognizing the structure to ease them on determining the amount. The recognition is started from the structure that is very close to them: their body structure. After that the structure can be broaden by exploring other structures, like: cards, dice, package or pile of something, and ice cube tray. Then using the structure and their experience on patterning, they would shorten their counting process.

Conjectures of the learning process. The teacher reminds the students about the activity of standing in the line and their drawing to model that situation. Then she can invite one child to stand, and question the others to determine how many eyes that child has. After that she invites two, three, five and eight children in sequence, and asks the similar question. The teacher could also question the reverse, for example, if there are eighteen eyes, how many children would there be. After that, she can rearrange the students in the group of four, and give the task for every group to determine how many eyes, ears, hands, feet in their groups. In this phase, the students would occupy the structure and pattern at the same time. Knowing the eyes or finger from each child was using structure, and knowing the total was using the pattern. The result of the investigation was recorded in the paper and will be presented and discussed during the class discussion. This part will be ended with the flash card game about the structure to emphasize students understanding.

Activity 4: Configuring object in structured and patterned arrangement

Starting point. We used students' achievement from previous learning as the starting point. We expected that in the previous learning students were aware of structures and could occupy them in their counting process.

The goal of activity. After doing this activity we expected that students would able to occupied structures in to represent a number and see the relation of the number to each other.

Intended activity. To involve the children on configuring the object in structured arrangement, we needed to choose the powerful number. The small number would not stimulate the students on doing structuring – instead, they could subitize. The bigger number was expected can growing the need of doing so. The students would make a configuration of object in a structured arrangement and do patterning to express a number.

Conjectures of the learning process. The lesson would be based on the game: *we have this much.* The children were grouped in pairs, and they were given some amount of manipulative object (e.g. bottle caps). The instruction would be started with make "5", "8", "10", …. The children were expected to occupy the structure from their previous activity – and maybe then patterning. Then the teacher could continue the instruction with "show me 17, 23, …". In this step, bigger numbers are involved to stimulate the emergence of structure and pattern in children's presentation. More challenge to structure, the teacher instruct the pair to play a role play on hiding some bottle caps and asking their partner to guess how many caps that are hidden. The students might find difficulty to handle the big and not proper number, namely 17 or 23. They might find difficult to find the structure and pattern of this numbers, because they would face object that could not be grouped in the structure that was being patterned. The students might perform better in ‘good’ number, like 9, 15, or 20. We also conjectured that even they could configuring the object on structured and patterned arrangement, on checking the amount they still used one by one counting.

Activity 5: Sorting the socks

Starting point. We would go back to students experience on pattern of twos activity (stand in line and packing) as the starting cognitive knowledge of the students on doing this activity. We also would occupy the fact that the socks were in pairs as the reality that would be mathematized during the lesson to come to the idea of double.

The goal of activity. After doing this activity, the students were expected to be able to use the idea of doubling to shorten their counting process. This idea of doubling would also be powerful in their later process of learning to offer flexible strategy for them to handle the arithmetical problems.

Intended activity. Using the fact that socks always come in pairs, children will explore one – to – one correspondence, grouping of two, and developing the doubling idea. Students are given a full box of socks and are asked to sort the socks and make inventory of the socks. They can sort it according to the sizes and/or the colors and note the quantity of the socks.

Conjectures of the learning process. Teacher started the lesson by showing a bunch of socks that was very messy. She would like to know how many socks in the bunch and asked the students to help her sorting the socks. Then she gave each group (on pairs) a box of socks and also blank sheet to record their inventory. During this activity we had some conjectures of the learning process of the students. (1) The students just counted the socks but did not sort them either in size or colors. We envisage that if the students still experienced this, it meant that they still did not have imagery from their learning process in previous activity. They did not have idea to occupy the structure of the socks, colors, or sizes, to ease them on doing the counting. They might need to recount when they were asked how many pairs they had. (2) First the students sorted the socks by trying to find the other pair. Then they count one – by – one, and count them again by pairs. When they experience this, we conjectured that they had an idea of one to one correspondence and could see the structure of the socks. They might not able to pattern the structures to ease them on counting process, but at least the students in this level were able to organize the objects in a structured arrangement. (3) First the students sorted the socks by trying to find the other pair, then count them by pair and determine the amount of sock without recount them one by one. We conjectured that the students in this group had already learned to advance their counting strategy – from one by one counting to skip counting by occupying the structure and the pattern.

Activity 6: Tens and units

Starting point. We used the students' familiarity about the structure and pattern in this activity. Then, as the learning environment that could be mathematize we would use the ice cube tray.

The goal of activity. After doing this activity, the students were expected to be able to group the objects into tens and units – as the preparation to learn about the place value.

Intended activity. In this activity, the students worked with the structure of tens. They observed the structure of tens in an ice cube tray. Then, using this structure and patterning this structure, students shorten their counting process by skip counting strategy.

Conjectures of the learning process. The teacher started the lesson by posing the problem about a party, and they needed to prepare the ice cubes for the drinks. She invited the students to observe the particular structure in that ice cube tray. After that she questioned students, e.g. if they needed 25 ice cubes, how to fill the trays with water? We expected that students see if in that ice cube tray there was "5 and 5" structure or "10" structure. We conjectured that since the numbers involved were 5 and 10, students would not find difficulty. It was possible that the students count one by one when they patterning the structure "5 and 5". But, patterning the "10" structure would be easier for them.

Activity 7: Taking inventory

Starting point. We used students learning experience about tens and units in the previous activity as the starting point.

The goal of activity. The goal of this activity was the same like the previous activity: the students were expected to be able to group the objects into tens and units – as the preparation to learn about the place value. We repeated this goal in aim to give stronger foundation to prepare the idea of place value.

Intended activity. In this activity, students would use their experience in the previous activity to group items in groups of ten. We expected that they saw the structure of tens and could relate numbers to each other by patterning and structure.

Conjectures of students' thinking. The teacher asked the students to make an inventory of the numbers of some items that she had prepared before. Children would work in pairs and group the items in group of ten. Then they registered the number in their inventory list. Here we expected that the students could see that 10 and 10 made 20, and add another 10 made 30, and so on. We conjectured that even though in the previous activity they had already experienced the tens and the units, on doing this activity, some of them would not occupy this knowledge to make group of tens. They might occupy another structure that was more familiar with them, e.g. structure of twos or fives.

As I said before in the beginning of this chapter, we only tried out some activity from this HLT and we decided to change this HLT with more appropriate activity. We say an activity is appropriate when the mathematical level of the activity can be coped by the students. I will present the try out result and the revised HLT in the following chapter.

5. Retrospective Analysis

Experiencing the cyclic process of development and research so consciously,
and reporting on it so candidly that it justifies itself,
and this experience can be transmitted to others to become like their own experience.
(Freudenthal, 1991)

This chapter is the last phase of our design research. In this chapter, first, I will present the try out result of our HLT on chapter 4 (from now and later we call this HLT as HLT 1). From this try out result, we – I, the teacher and my local supervisor – realized that HLT 1 is not appropriate to support students' development of number sense. Thus, based on the findings and our discussion, we proposed the second HLT (we called it as HLT 2). The teacher contributed on this refinement by giving advices based on her experiences on teaching the grade 1 on that school. We tried out some activities of the HLT 2. But, due to the time limitation and the school condition, we could not try out the whole HLT. We analyzed and discussed the result to make some improvements on this HLT 2. We called this improved HLT 2 as HLT 3. We used this HLT 3 in the real teaching experiment. Then, I will present the result of the teaching experiment and also the analyses in the following section. The last section of this chapter is the conclusions of our analyses. In the conclusion I will answer my research questions.

5.1. Testing the HLT 1 and the interview to refine the HLT

On May until June 2008, I tested the HLT 1. First I discussed HLT 1 with the teacher of the first grade. After she read the outline of the HLT 1, she commented on some activities that might not be able to be conducted in the real teaching experiment that involved 20 children. She also reminded me that the subject in the real teaching experiment were the new first graders. And according to her, looked back to her experience on teaching those new first graders, the new first graders in that school were the students who never went to the kindergarten before. They might have difficulty on counting and writing. Thus, she concerned on some activities that involved the big numbers (e.g. shorting the socks, ten and units, and taking inventory). Then, we tried to test some activities from this HLT. But due to the time (at that moment the school was busy to prepared the students for annual exam), I could only conduct the testing in a small group 2 to 10 students.

Stand-in-line context problem. This context was chosen because I observed that every day the students have to stand in line before they enter the classroom. When the students were given the task to make a good arrangement of some number of students, the result showed that there were many interfere from external their norm, i.e. the fact that they need a leader who has to stand in front of the row and that the number of the rows could not be fair. I tried out this activity to ten students about the stand in lines context problem, using *pions* as the model of the students. Then they started to discuss the arrangement. They made line became 13 and 14, and 1 child stand in front as the leader. Another issue became apparent when it was observed that the two lines that they made were not in one to one correspondence. Then the pattern of twos was not appeared. I also observed that on patterning the 2, students were not fluent to mention the numbers. Since this activity was the first activity in the HLT 1, we (I and the teacher) were afraid that the new first grader would not able to cope the mathematics level. They might still have difficulty to recite the number, but we would ask them to count by two.

Body-part context problem. This context was chosen in the reason that the body-part, i.e. eyes, ears, hands, feet, etc, are in pairs and it always the same for each people. The children were asked to count the amount of, namely, the eyes of a group of people. They also were asked to make a group that consists of, namely, ten ears. On proving the group they made was right, they need to count. Some children occupy the structure and using skip counting strategy, but some also did one by one counting. This also yielded the same anxiety as the previous experiment, that they might even were not able yet count in good manner, while this activity demanded the ability of advance counting, that was skip counting. The social norm also held a big influence in this activity. For example, when they were asked to arrange a group, instead of joining the group, some children only wait for the other fellow to come. Or the group of girls would not join with the group of boys, although if they willing to join, they could make a good group.

The teacher really suggested that the two-third of the activity engaged the problem with small number, i.e. l0 for the maximum. Then we started to discussed about another kind of activity that more proper with students' level. I asked her permission to interview some students first before developing another activity.

These *interviews* were done either individual, in pair or in group. All of the interviews were task interviews where the students need to show actions instead of writing or giving short answer. The interviews attempted to dig up the background of learning and mathematical ideas that children already have in their mind. There were two purposes of these interviews. The first purpose was aimed to know what kind of strategies students used to solve the problem related with counting. And the second purposed was aimed to know what kind of context that are familiar with the children and could be occupied as the structured context.

To know the kind of strategies students used on counting, students were given two kinds of task. The first kind was tasks related with unstructured object. Students were given some unstructured object. First they were asked to count how many objects they had and then they were asked to show some amount of object. The result showed that in doing the task, most of them counted the object one-by-one. They tagged the object while reciting the number word. Some of them could synchronize the tagging and the word, but some still were not able to. Thus, it did not match between the number of object and the result of counting. It seemed for them, the instruction of counting meant that they should do one-by-one counting, either by tagging the object or observing by distance. Observing the class incident related with counting, it could be explained that this insight came from the norm they had in the class. All the time when the teacher asked them to count something, they had to say it aloud one-by-one, while the teacher tagged the object. They did not feel that they need to use other strategy, e.g. grouping and arranging the object, to identify the amount of object.

The task interview then was refined with “Take away and guess” game. They were given some amount of object. They were asked to remember the amount of object they had then they were asked to close the eyes. The researcher took away some object, and asked the student to guess the number of the object that had been taken away. It was expected that they would start to arrange the object in such a way so that they can easily count the number of the object that were missed. As the game played, most of them did not have initiative to manage the objects. In this happening need the researcher to give remark as an input to develop the didactical phenomenology.

The second kind of task was tasks related with structured object. The students would be observed whether they familiar with structured object and could deal with it in the counting task. There were three tasks. The first task was started with counting the unstructured object then they were given help to organize the object. The second task was counting with arithmetic rack and pattern of ten. And the third task was done as the class activity. The class played pictured card game. First, they were showed picture-card with structured object and asked to show the correspondence number card. Second, they were showed number-card and asked to show the correspondence picture-card the amount. The result showed some important remarks. First, students were not handy with structures and pattern. In their learning experience they did not meet structure and pattern very often. Second, the students could occupy the structures and pattern after they became familiar with it. It was thought that to make a structure familiar to the students, then it should come from the students' reality.

Having the result of the interview, I proposed another sequence of activities using the butterfly wings as the contextual situation. As when we see from the mathematic point of view, the butterfly wings have a very interesting fact. The both side of the wing of a butterfly are patterned and the pattern of the left side and the right side are symmetrical. From the Indonesian students' interest, butterfly is very attractive. They used to play in the garden with the butterflies or even caterpillars. They experienced and had seen that there is symmetrical property in the butterflies' wings.

We can mathematize this contextual situation in many ways. The first one, the dots on the wings were in a special arrangement. Thus we can use it as the motivation for the students to make dots arrangement on the butterfly wings to represent the small numbers. The second one, the wings are symmetrical. Thus we can use it as the motivation for the students to patterning. And to convince us that this reality can be experientially real for the students, we tried out two simple activities related with the butterfly.

Butterfly wings contextual situation. First, we questioned the students to draw a butterfly. From their drawing we saw that all of them draw the symmetrical pattern in the butterfly wings. It was evident that the symmetrical pattern on the butterfly wings has become their common sense. Second, we questioned the students to predict how many dots in the butterfly wings if we know that on one disc we have, for example, three dots. Either used the counting one by one or the doubling, the students could say that it was six since the butterfly wings had the same amount of dots on the right side and on the left side. It supported my prediction that students had a common sense about the number and the symmetrical pattern on the butterfly wings is experientially real for them.

5.2. HLT 2

My goals of this sequence of activity are supporting children's perception about number and their counting ability as a part of the development of number sense. On giving this support, I elaborated structure, pattern, and symmetrical pattern. I developed the activities that helped students on perceiving the numbers by having the images and the meaning of the numbers. At the same time, using these activities, I also tried to support students' development on counting strategy. I evoked the idea of doubling, almost doubling, and skip counting as an alternative for one-by-one counting. Having done the activity, students may see the relation among numbers in the concrete and abstract level.

I tried to develop the instruction in realistic approach. Thus, the activities in my instructions would be meaningful for the students and meet with their common sense. I began with observing the butterfly wings. As I have mentioned in the previous section, when we see from the mathematic point of view, the butterfly wings have a very interesting fact. The both side of the wing of a butterfly are patterned and the pattern of the left side and the right side are symmetrical. From the Indonesian students' interest, butterfly is very attractive. They used to play in the garden with the butterflies or even caterpillars. They experienced and had seen that there is symmetrical property in the butterflies' wings. They had belief that butterfly wings are always symmetrical, although in their daily life they would use the term "the same" on mentioning this property instead of symmetry. Mathematically, these two terms are not the same. The structure of my right hand and the right hand of my friend are *the same*, but the structure of my right hand and my left hand are *symmetrical*. Nevertheless I accept that as the starting point of the children.

I found that this contextual situation was powerful as a mean to reach our endpoints. The pattern on the butterfly wings can motivate the students to make a configuration of objects in a good arrangement to represent the numbers. And by having experience of this arrangement object, the children would develop the image of the number and could relate numbers with other numbers and their world. The symmetrical property of the butterfly wings would encourage the students to see the double and also the almost double. Seeing the pattern on the one disk of the wings, children could predict the pattern on the other side since they are symmetrical. I envisage that this hypothetical learning trajectory would work out in the following manner.

a. **Activity 1**: Observing the butterfly wings

Starting point. The starting point for doing this activity is students' common sense about the symmetrical property on the butterfly wings. The contextual situation about the symmetrical pattern on the butterfly wings would be mathematize during this activity to come to the awareness of the symmetrical pattern.

The goal of the activity. By doing this activity, students become aware of the symmetrical pattern on the butterfly wings.

Intended Activity. The sequence started with investigating pattern in a butterfly wings. We envisaged that by investigating patterns on the butterflies' wings and looking at its characteristic, students would gain the idea about symmetrical in an understanding. The students would discuss the pattern on the butterflies' wings, compare those patterns, and find the resemblance among those patterns – that is the symmetrical property. The students might use various words on discussing the phenomena as the term “the same” or “similar” or “symmetrical”. We expected that even though the students did not mention about symmetrical, they could recognize that the right side of the butterfly wings was symmetrical with the left side.

Conjectured of the learning process. The teacher showed some picture of the butterflies, and asked the students to observe the pattern (dots, lines, strips) in the butterfly wings. She invited the students to compare the left side and the right side of the wings and also to compare one butterfly to others butterflies. The aim of this discussion was that student getting more able to recognize the pattern and starting to have the idea of similarity. Students do not have to use the word pattern when they do the discussion. In this phase of learning it was enough for those young learners to get only the idea. They may use the more informal word, like: the same. After the class come to the agreement that the butterfly wings have particular pattern and they are similar in the left and the right side, the teacher can move to the next step of the learning line. The teacher asks the students to make their own butterfly wings. We prepare the butterfly-shaped papers with coloured pencils and distribute them to the students. The results of students’ works were displayed in the classroom wall. The teacher invited the students to tell about their own butterfly wing and also to comment to others.

b. **Activity 2**: Configuring dots on the butterfly wings

Starting point. Using their awareness of the symmetrical property on the butterfly wings from the previous activity as the starting point, in this activity, students started to make arrangement of dots on the butterfly wings. We also involve their ability to make one to one correspondence and also to subitize as the starting point.

The goal of the activity. By doing this activity, students tried to represent the number by configuring the dots in the symmetrical pattern.

Intended activity. During this activity, the students made dots configuration to represent a number that was being asked and reason using structure or the symmetrical pattern. On arranging the dots, students might do that in symmetrical pattern since they have already the imagery of learning from the previous activity. They discussed in paired about the possible arrangement of the dots to represent a number.

Conjecture of the learning process. The teacher would start to demand an even numbers. We envisage that an even number would enable the students to use their understanding of symmetrical pattern during configuring the dots. “Four” would be a good choice, since “four” is a small number but give more possibility to the given answer than “two”. Then continue with 6, 8 and 10. It was predicted that the students will put the dot evenly in both side of the wing. But it was also possible that may happen a child does not put the dots evenly. After discussing the even, the teacher could start to demand the odd numbers. The odd could be very difficult since to make it keep symmetrical, students needed to decide putting at least one dot in the middle. By putting the dot in the middle seems that student understood the idea of half to make it fair. Half part for the left side, half part for the right side. The students might not aware with this understanding, but they could use it to solve the problem. We conjectured that (1) students would put one dot on the one side, then one dot on the right side, simultaneously to get the expected number; (2) students would pattern the dots by putting first some dots on the left and the same amount of dots on the right, then thinking about the rest.

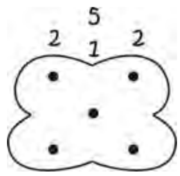
c. **Activity 3**: Mini lesson about dots configurations to express a number.

Starting point. We used students’ experience from second activity as the starting point: configuring dots in the symmetrical pattern to express a number.

The goal of the activity. The main goal of this activity was the same as the previous activity: students could represent the number that was being asked and reason using structure. We repeated this goal in the classroom community to give chance for the students to share their idea and take the idea from others as a contribution on developing their own idea. The second goal was that students could start seeing the numerical pattern to express a number.

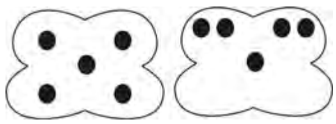
Intended activity. In this mini lesson activity, students needed to configure the dots in the butterfly wing as many as the number demanded. By numbering each part of the wings, it was expected that the students started to see the numerical pattern to expressing a number.

Conjecture of the learning process. The students presented their work on configuring dots on the butterfly wings to represent a number. The teacher guided them to number the wing that represented the number of the dots in the left, in the middle, and in the centre, and also the total. They numbered the wings in the reason that they needed to know the kind of the butterfly wings they had. The students might group the dots in different ways, i.e. in horizontal way instead of vertically (left, middle and right). The students needed to have a discussion about the good way to numbering the butterfly wings so that they could easily differentiate one wing to another and could compare those wings. In the later lesson this number was very powerful to guide the students to see the relation between numbers. For example, one possible arrangement for five dots was



The “5” was the total of all dots, the “2” in the left was the number of dots on the left wing, the “1” in the middle was the number of dot on the middle of the wing, and the “2” in the right was the number of dots on the right wing.

We predicted that the students would only focus on the dots configuration in the symmetrical pattern and could not see the numerical pattern on the butterfly wings. For example:



Since visually it looked different, students might recognize these two configurations as the different things, although when they put the number on it, it would produce the same numerical pattern (or relation)

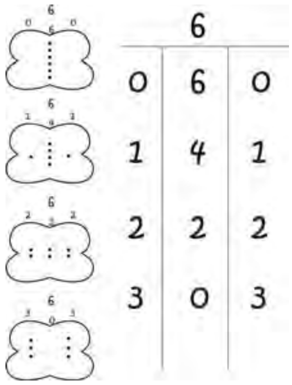
d. **Activity 4**: Record all possible configuration

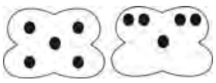
Starting point. We used students experience from activity 3 as the starting point for this activity. In the previous activity, the students saw various ways to make dots configuration on representing a number.

The goal of the activity. The goal for this activity was that students could see the relation of number to each other and also the realizing that there were many ways to represent a number.

Intended activity. First students would make dots configuration on the butterfly wings. They were asked to make as many as possible configurations that showing the same number. A table was introduced to guide the students to see the relation among the numbers. Using this table, students were guided to move to the more formal level.

Conjectures of the learning process. The students would collect all possible dots configuration for a number, and arrange them in one table.



The students discussed in pairs to record all possible arrangement of dots on the butterfly wings for a certain number. In this activity, students would be confronted with a dilemma to decide whether  were different arrangement or not.

e. **Activity 5**: Perching butterflies

Starting point. The starting point of this activity was the students' awareness about the symmetrical pattern on the butterfly wings (experience from activity 1 and 2). We also took the idea of the perching butterfly from the daily life. When a butterfly was perching, most of the time we could only see one side of the wings.

The goal of activity. The goal for this activity was giving support to the students about the idea of doubling and almost doubling. In the later learning process, the doubling and almost doubling idea would be a powerful strategy to solve the arithmetic problems.

Intended activity. In this activity students were invited to look back on their experience on activity 1 and activity 2. Then they discussed about the possible number of dots on the butterfly wings if they could only see one side of the wing. They might use their learning experience about the awareness of the symmetrical pattern on the butterfly wing.

Conjectures of the learning process. The teacher started the lesson by telling a story about the butterfly which was perching on a leaf so that they could only see one side of the wing. This story became the central discussion during this activity. They were asked to determine the amount of the dots on the both side of the butterfly wings. It was expected that by discussing this problem the students came to the idea of double and almost double. The students discussed the method to determine the amount of the dots in the both wings. In this activity student might also discover the strategy of counting on. They knew already the amount of the dots on the one side. Thus to know the total they recounted the visible dots. There were three conjecture of students' thinking in this activity. The first one was when they struggling finding the answer by counting the dots two time. So, to tell the number of the dot on those butterfly wings they had to sum the number of the dots in the two sides. The second one was when the children captured the number of the dots on the butterfly wing at once (by subitizing or recognizing the structure) and counting on the rest dots to get the total. In these two incidents the students might do this without realizing that they did the addition. The last one was when the students still did not have the idea about symmetric or doubling. These children may only write the number of the dots that they see.

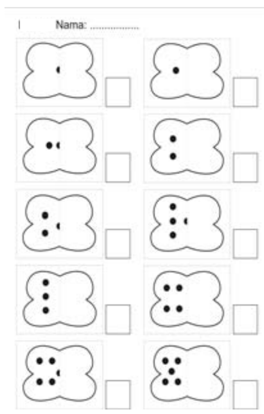
f. **Activity 6**: Complete the dot configuration

Starting point. For the starting points to do this activity, we used students' experience in the previous activity and students' awareness of symmetrical pattern on the butterfly wings.

The goal of activity. The goal of this activity was developing the idea of double and almost double.

Intended activity. In this activity, first the students would work individually. They would complete the drawing of the dot configuration on the butterfly wings. In doing this activity, they needed to pattern the dots by repeating the dots configuration in the left side. Then they would share their idea in the class discussion about their strategy on accomplishing the task.

Conjectures of the learning process. The students were asked to complete the wing by making a drawing of dots on the other side of the wing.



There were three conjectures of students thinking in this task. The first one if the students really got the idea of symmetrical. Then they would not have any difficulties to complete this task. They would pattern the dots to get the good picture of butterflies. The second one when they already got the idea of symmetrical but they do not know how to handle the half dots, they would not accomplish this task successfully. Some children might see this half as one complete dot and did not realize that it was a half, thus they mirrored this half dot and they got one extra dot. Some children might see this half dot also as one complete dot but they did not mirror this half dot since it stayed in the middle of the butterfly wings. The third was when the children do not have any idea about the task.

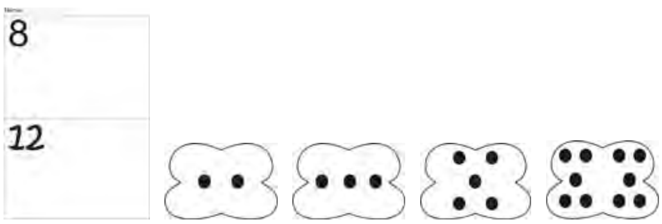
g. **Activity 7**: The flutter.

Starting point. In this activity, the students' interest moved from working with a single butterfly to seeing butterfly as a group. We would use the phenomena that the butterflies sometimes fly in a group (It is somewhat poetic and apropos that a group of butterflies is called a "flutter"). A flutter would consist of the same species of butterfly (it meant all the butterflies in a flutter had the same dots configuration on their wings). From this situation, the teacher would guide the student to mathematize the situation to come to numerical pattern.

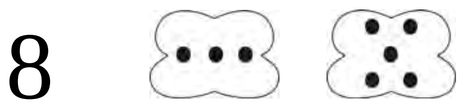
The goal of activity. The goals in this activity were that students could relate, compose, and decompose numbers by patterning.

Intended activity. Students would arrange the butterfly wings in such a way that the total amount of dots on the butterflies' wings fulfilled the demanded number. During sharing the idea in the class discussion, we expect that the students would start to see the patterned dots that representing a number.

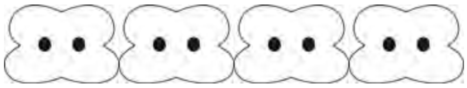
Conjectures of the learning process. There were four kind of dots arrangement in the dotted butterfly: dotted butterfly with two dots on it, with three dots on it, five dots on it, and ten dots on it.



There were two conjectures of students thinking on fulfilling this task. The first one was when the students think that they could use any number of the dotted. For example, to fulfil the 8, they stuck butterfly with 3 dots and 5 dots.



For this group of student, we would say that they did not use pattern to support their strategy, but they used the number relation. The second one was when the students decide to use patterned dots to fulfil the number. For example, to fulfil the 8, they stick four times 2 dots butterfly.

8 

There might two different strategies that students use in this activity. The first one was when the students tend to use trial and error strategy. They picked up the wings, count the dots then glue the wing. After that they counted the dots again. When it was okay then they moved to the next task, but when it was not okay then they unglued the wing and changed with other wing. They might do this several time until they get the answer. The second one was when the students well plan the wing thus they did not need to pick and change many times. Usually it happened to the students who chose the patterned number.

h. **Activity 8**: Numerical pattern

Starting point. The starting point for this activity was students' experience in the previous activity.

The goal of activity. The mathematics goals in this activity were that students could relate, compose, and decompose numbers by using the numerical pattern. Thus in this activity, students would work in a semi formal level.

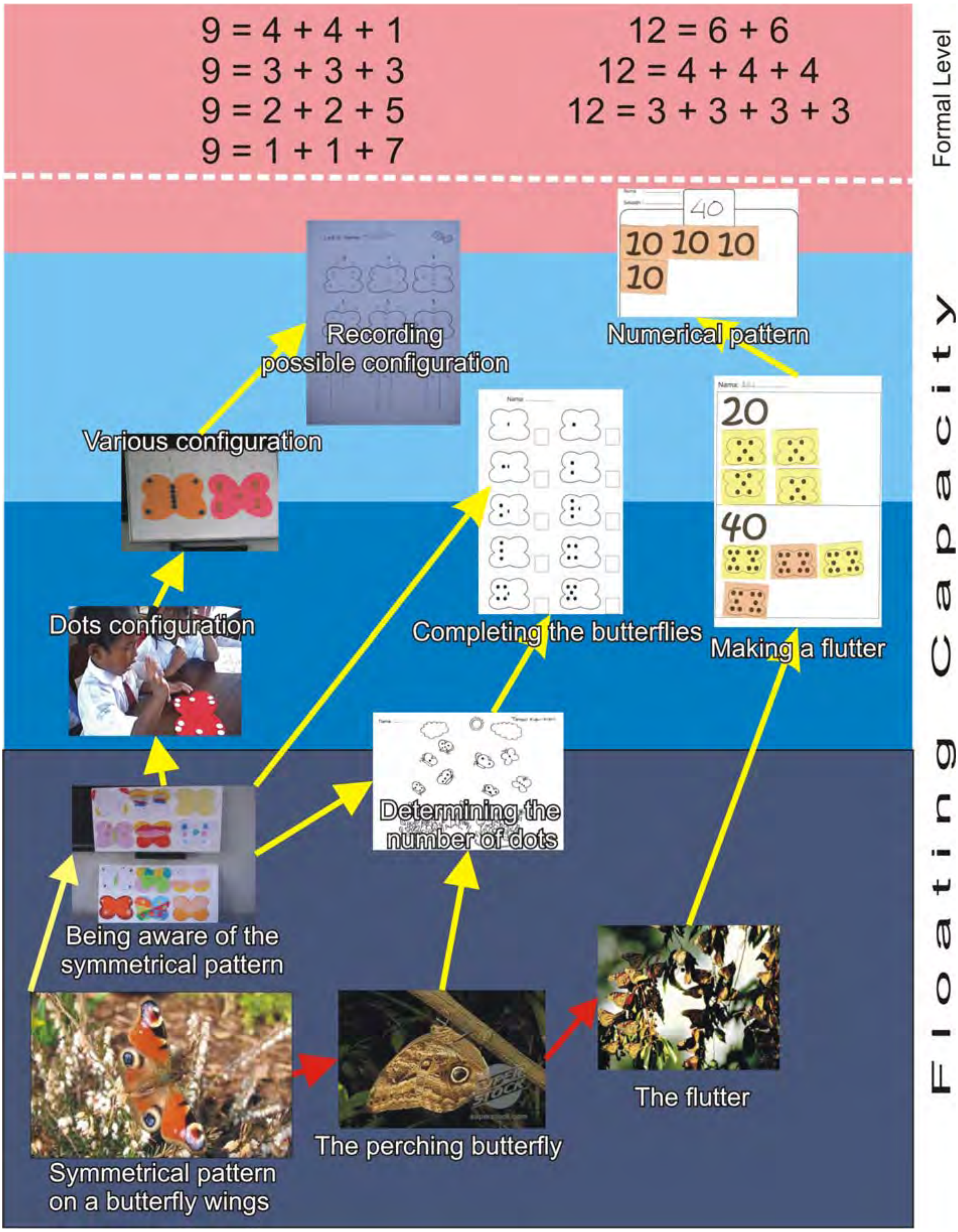
Intended activity. The dots in the butterfly wings would be replace by number that represented the amount of the dots. Students would arrange the butterfly wings in such a way that the total number fulfilled the demanded number.

Conjectures of the learning process. The task was still the same, that was making a group of butterfly wings which the total amount of the dots on their wings fulfilled a demanded number. The difference was in this activity students were stimulated to move to the formal level. The dots in the butterfly wings were replaced by the number represented the amount of the dots. The shift from working with dotted wings to working with numbered wings was related to the model of – model for transition. Initially, the students' work with the model would foster the framework about the number relation. After the transition, the model would become a model for generalized mathematical reasoning.

In summary the skeleton of sequence for supporting children's number sense development would be explained as the following.

1. *Being aware with pattern on the butterfly wings.* Using their common sense about the symmetrical property in the butterfly wings, students started recognizing special arrangement in the butterfly wings. The mathematical practice that we expected to emerge was the students could express the numbers using symmetrical pattern. The students discussed the phenomena using the term "the same" or "similar" or "symmetrical". Then the students started structuring the dots in the symmetrical pattern to express the quantity of a set of dots on the butterfly wings.
2. *Various dots configuration to represent a number.* The students proposed various kind of dots configuration to express the particular number. While doing so, the students developed a framework about various representations for a number and the relation of numbers to each other.
3. *Recording the configurations that represent the same number.* Using a table, students would organize the various representation of a number. This type of activity would offer students the strategy to shorten the counting process by recognizing the relation among the number. By recognizing this relation, students could see the relation of numbers to each other and built a preliminary foundation of double and almost double idea.
4. *Applying symmetrical pattern to develop the idea of double and almost double.* The students developed the fundamental methods for arithmetical reasoning based on the framework of number relation by the idea of double and almost double. We believed by being familiar with the idea of double and almost double, students would develop strategies to solve arithmetical problem in their upcoming learning process.
5. *Patterning to develop the skip counting strategy.* The students would reason about the composition or decomposition of a number by seeing the relation of number to each other. Students will also develop the strategy of skip counting.

In the following figure I tried to wrap up the HLT that I have presented above. I drew also the formal level of that learning line, but actually in this HLT I did not explore the formal level.


9 = 4 + 4 + 1
9 = 3 + 3 + 3
9 = 2 + 2 + 5
9 = 1 + 1 + 7
12 = 6 + 6
12 = 4 + 4 + 4
12 = 3 + 3 + 3 + 3
Formal Level
40
10 10 10
10
Numerical pattern
Recording
possible configuration
Various configuration
Dots configuration
Completing the butterflies
20
40
Making a flutter
Determining the
number of dots
Being aware of the
symmetrical pattern
The flutter
The perching butterfly
Symmetrical pattern
on a butterfly wings
Floating Capacity

The Cascade of Tools and Imagery

The first contextual situation that would be mathematize was the symmetrical pattern on the butterfly wings. And the first tool that was used was picture of butterfly wings. This tool was used to give meaningful insight for the students and invite students became aware about their common sense of symmetrical property on the butterfly wings. The second tool, butterfly wings and the dots model was used to keep the insight in students mind when they were doing the activity. The next tool was a table that was introduce by the teacher. This table would help the students to see the relation of number to each other. This relation in the later activity would be developed as the idea of double and almost double. The perching butterfly would be the next contextual situation that would be mathematize. From here students would develop the idea of double, even though in my activity I did not go until the doubling it self. Te last contextual situation was the flutter (a group of a butterfly). While mathematizing this situation, the students would develop the skip counting strategy.

This succession of tools assured us about the comprehensiveness of the activity. The cascade of imagery were hoped to be a useful way to describe how the proposed sequence of tools can be seen as reflecting RME's theoretical reinvention process.
The table bellow shows the summary of the role of tools that we proposed.

Tool/ Contextual Situation	**Imagery**	**Activity/ Taken-as-shared interest**	**Potential mathematical discourse topics**
Symmetrical pattern on the butterfly wing	-	Awareness of symmetrical pattern on the butterfly wings	Pattern, symmetrical
Model of dots and butterfly wings	Symmetrical pattern	Configuration of dots to represent a number	Dots configurations to represent a number, pattern, symmetrical
Table	Dots configurations to represent a number	Collecting various representation of number	Various configuration to represent a number, pattern, symmetrical
Perching butterflies	Symmetrical pattern	Double and almost double	Double and almost double, symmetrical
The flutter	Pattern	Patterning numbers	Skip counting strategy, numerical pattern

5.3. Retrospective analysis of HLT 2 and HLT 3 as the refinement of HLT 2

On June 9 and 10, 2008, we tried out the outline of HLT 2. We began the activities with recognizing structure and arranging the dots on the butterfly wings. On this try out, we asked the students to make drawing of the butterfly. As it has predicted before, the students had a common sense about the symmetrical property on the butterfly wings. The wings of the butterflies they drew were symmetrical. The right side and the left side of their wings have the same pattern. Then we move the activity of configuring the dots on the butterfly wings. We asked the students to draw the dots on the butterfly wings as many as the number that was demanded. Students did not find any difficulty to draw the dots to represent the numbers. The interesting observation was when they drew the dots. For example, when they were asked to make a butterfly wings with 10 dots, they drew first five dots on the left side of the wings then the other five dots on the right. It seemed for us that those students already knew that 10 is made from 5 and 5, then they have first put 5 in the left and another 5 in the right. These students using pattern to represent a number (repeating 5 and 5). It would be different if they first put one dot in the left then one dot in the right, and another dot in the left then another dot in the right, and so on until they got 10 dots. These students occupied the symmetrical idea, but actually they count one by one to fulfil the task.

In the activity to enact the doubling, we showed to the students the one side of the butterfly wings. Then we questioned them about the total dots on that butterfly wings. Students told their idea on how to determine the total amount of the dots. They said they had to multiply it by two since the other side of the wing had the same amount of dots. The strategy students used in this activity was higher that our expectation. It was reasoned that the students in our tested group were students from higher level. The students in our next teaching experiment were the new first grader, thus we could not expected that they would give such strategy. Even we had to be aware and tried to give more support for these new students to be able to reach the expected level.

After we finished with tested the HLT 2, we made a discussion about the HLT 2. The teacher suggested that we needed to add an activity to bridge the shift from working with one butterfly to the group of the butterfly. We thought that the gap of mathematical concept from the idea of double and almost double to the skip counting was too far. Then we decided to add two activities that we expected could bridge the gap.

1. *Pairing the wings*

Starting points. We added this activity before the activity 5 in HLT 2. Therefore the starting point of the activity 5 became the starting point of this activity. Then the starting point of the activity 5 would be the students' learning experience from doing this activity.

The goal of the activity. The goals of activity were to support the awareness of symmetrical pattern and develop the idea of double and almost double.

Intended activity. In this activity, students would discuss about the pair of the butterfly wings. We expected that they could reason by using the symmetrical pattern to determine the right pair.

Conjecture of the learning process. The teacher started by telling the story about the butterfly which was perching on a leaf so that they could only see one side of the wing. Then the students discussed to determine the other side of the butterfly wings by finding the pair. During the discussion, the students need to aware about the symmetrical pattern and half dots. They need to see they need another half dot to make a half become a dot.

2. *The flutters of the butterfly*

This additional activity was added as the result of our reflection that we need to bridge the gap between working with one butterfly to working with the group of butterflies. We put this activity before the activity 7.

Starting points. Since this activity would be put before activity 7, thus the starting point of the activity 7 became the starting point of this activity. Then the starting point of the activity 7 would be the students' learning experience from doing this activity.

The goal of the activity. The goals in this activity were that students could relate, compose, and decompose numbers by patterning.

Intended activity. During the mini lesson students would share their idea to determine the total number of dots on in a flutter. They would propose the shortest way to determine the total dots. It was expected that students would come to the skip counting strategy.

Conjecture of the learning process. The activity was a mini lesson about group of butterflies which has the same amount of dots on their wing. The teacher asked the students to determine the total amount of the dots on those butterflies in that group. Students would discuss the easiest and the fastest way to determine the total. We expected that by patterning, students could come to the skip counting strategy.

5.4. Retrospective analysis of the teaching experiment

As it has been mentioned above, the experimental subject was the new students in grade one. The subjects were in age 5 – 7, and some of them have already had an experience in kindergarten. This was different with our prediction that they did not go to kindergarten. In each lesson not all of the students presented in the classroom. From 21 students registered, the maximum number students available in the classroom are 20.

In this subchapter we will present the retrospective analysis of the teaching experiment. We will divide our presentation in three sections. In the first section, we will present the result of our pre test. This pre test was aimed to know the starting point of the students. In the second section, we will present about the observation and the retrospective analysis of the teaching experiment. As we have seen in our HLT, we made five phases mathematical practice in the sequence of the development of the number sense. On reporting the teaching experiment and the retrospective analysis we will discuss each phase of the sequence. In the last section, we will present the result of the final test. We use this final test to find evidences on students' learning process.

5.4.1. Pre test as the information of the starting point of the students

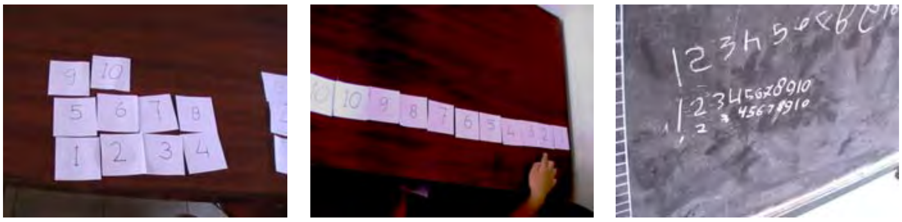
This pre test was held on July 15th and 16th, 2008. As it has been mentioned before the aims of our pre test were to see the ability of counting, the meaning of counting for the students and the strategy they had and also their knowledge about the formal symbol of number.

On the first day, the teacher invited the students to get the insight of number by singing a song about numbers. Then she questioned whether the students know the other number. The students started to mention some numbers. Then the teacher asked the students: we are new here. Do you know how many students in this class? The students then directly counted by pointing each student. They related each student with one number. Then they came up with some different answers: 18, 19, and 20.

Here we see that the instruction of "Do you know how many …" for children means that they have to count. The teacher did not mention the word count. But for them it seemed that they have to count to know how many of something. And for these students counting meant pointing and relating the each child with each number. The children might come up with different answers since some of them either did not synchronize the pointing and the one to one relation or they skip some children in the class.

The teacher then questioned the students to count only the boys and then only the girls. This time the teacher used the term 'count'. The children gave the same respond. They count by making one to one correspondence between the set of boys and the set of numbers, and also between the set of girls and the set of numbers. They claimed the last word they mentioned as the result of counting. We observed that on doing the task the students did not use other strategy than that. The students did not distinctly separate the boys and the girls to count them easily. The teacher then asked the students to write the number represented the number of boys and the number of girls. One student wrote 9 for the boys. Another student wrote 10 for the girls. Both of them wrote the right number.

The teacher then distributed the number card and asked the students to make good order, started with the descending and then the ascending order. Most of the students did the task well. Only one student did not do the task. It could be that either he did not understand the task or he did not interesting with the task. After the students had the order, the teacher started to ask them to write the numbers in the blackboard.



Some of students' works

Looking back to the aim of the pre test for the day 1, we could make some conclusion that the students have been able to count by corresponding the set of object and the set of the numbers. The students have also been able to order the number and write the symbols of number.

On July 16th, the pre test continued. This time, our aim was to know how the students react to structures and how they perform the structure on counting. The teacher distributed the number card again. Then she told the students that she was going to show an amount using her finger and asked the students to show the correspondence number cards. In this pre-test we intentionally use the structure of the fingers. We believed that the structure of the fingers was easily recognize and familiar to the students. They should know that each hand consists of five fingers. We expected that the students did not do one by one counting for each fingers. They might just recognize the quantity at glance or they might do the counting on. The result showed that the some of the students either do one by one counting or just recognize the number and could show the right number card.

The task then reversed. The teacher distributed to the students the playing card without numbers. The arrangement of the icons in the playing card is recognizable. Might be that some students did not really familiar with the arrangements. Thus, first the teacher asked the students to order the cards. It was expected that by ordering the cards students knew that there was a certain arrangement of the icons. After that, the teacher showed the number cards and asked the students to show the correspondence playing card. We will discuss some students' work in the following part.

Observation 1 on Iksan's work: Ordering the playing card.

There had been several cards on the table. Iksan was holding the card with seven clovers.

- He pointed to each clover while recited the number: one, two, three, four, five, six, seven. Each number for each clover.
- He shouted: seven
- He looked in to a card, and pointed the clovers in that card while recited: one, two, three, four, five, six, seven, eight.
- He shifted the cards in the left side of the card with eight clovers.
- He put the card with seven clovers in the left side of the card with eight clovers.

From this example we saw that Iksan knew about the quantity, he compared the quantity and he ordered the quantity. But looking back to the aim, we saw that Iksan did not use the structure in his process of acquiring the quantity. It might be happened as Iksan did not recognize the structure of the clovers in the playing cards.

Observation 2 on Iksan's work: Helping Ari to order the playing card.

Ari did not complete the task until Iksan and Wihas has already finished theirs. Iksan (I) and Wihas (W) helped Ari (A) to order the cards.

- I: Took the card with two clovers and put it in the middle of the table.
- I: Took the card with three clovers and put it next to the two. While doing so he was looking to his own.
- I: *Shouted: four*
- I: Recognize the four then he took it and put it next to the three.
- W: Put the five next to the four.
- I: Saw his own work, pointed his own card, and said: two, three, four, five.
- I: Pointed to Ari's card, and said: two, three, four, five.
- W: *Said: now six*.
- I: Took the six, put it next to five.
- W: *Said: that's seven*. Pointed to the seven.
- I: Took the seven, put it next to six
- I: *Said: eight.* Took the eight, shifted the eight next to the seven.
- I: *Said: nine.* Took the nine, shifting the nine next to the eight.
- A: Pointed to each clover in the eight, recited: one, two, three, four, five, six, seven, eight, while tagging each clover.
- A: *Said: this is eight.*
- I: Pointed each card while *saying: two, three, four, five, six, seven, eight, nine.*
- I: take the ten; put it next to the nine.

It was interesting to see that while helping Ari, in the beginning Iksan tried to copy his own work. He arranged the cards and looked at his own. He tried to find the reference to finish the task. Then Wihas started doing the intervention by putting the five, pointing the seven and eight. Wihas did it without copying any work that had been done before. Iksan then recognize the eight and the nine without copying his work. It seemed that Wihas gave the influence to Iksan, said by his action that they did not to copy any work. Another interesting point was in this observation Iksan did not do one by one counting anymore. He pointed and arranged the cards by recognizing the amount.

Observation 3 on Iksan's work: Choosing the right card

The teacher showed the number card with 8 on it.

- Iksan tagged his card one by one in an order from the left to the right. *Said: two, three, four, five, six, seven, eight.*
- *He shouted: eight.*
- He picked the eight.
- He tagged Ari's card one by one in an order from the left to the right. *Said: two, three, four, five, six, seven, eight.*
- He looked at his eight card. Pointed each clover in the card. Said: *one, two, three, four, five, six, seven, eight.*

In the beginning Iksan did not count the clovers in the card. Instead, he used the fact that he had already order the card. He believed that he made a good order from two to ten. Thus to find the eight, he only need to choose the seventh. But then he evaluated his own work by counting the clovers in the card that he had picked up. In this episode we saw that Iksan tried to control his own result.

Observation on Nanda's Work

The teacher showed a number card with 5 on it.

- Nanda chose a card, and counted the clovers on the card. *Said: one, two, three, four, five, six.*
- She moved to the card in the left. She counted the clovers on the card. *Said: one, two, three, four, five.*
- *She shouted: five.*
- She chose the card.

Nanda started choosing any card then she counted the clovers. She found that the amount of the clovers was not 5 but 6. She chose the card in the left side of the previous card. She might know that card with six clovers was too much. She might remember that she had order the card from the fewer amounts to the more amounts. Then to have card with five clovers she went to the left. But she did not decide directly that the card next in the left was card with five clovers. She needed to count the clovers. Compare what Iksan and Nanda did we saw that they acted different. Both did not occupy the structure of the clovers. Iksan used his experience from the previous learning experience, that he ordered the cards whilst Nanda tried to guess the card.

Conclusion

Back to the aims of this activity, we can see that the students did not see the structure and did not try to use structure on doing the task even though the objects were structured. It means that the first need for these students is recognizing structure. They need to see that there was a special arrangement of the object. Then they need to see the relation between the arrangements and the numbers. We also see that students were in the beginning of knowing and using the number symbols.

5.4.2. Retrospective analysis for HLT 3

As we have mentioned above, on reporting the teaching experiment and the retrospective analysis we will discuss each phase of the sequence. We will start with phase one where the students came to the awareness about the symmetrical structure on the butterfly wings. Then we will continue with the phase where the students develop the framework about various representations for a numbers and the relation among the numbers. After that we will present the phase where the students organized the various representations of the numbers to get the better comprehension about the relation among the numbers. Before we present the last phase where the children developing the strategy of skip counting, we will present the phase where students develop the fundamental methods for arithmetical reasoning by the idea of double and almost double.

a. Phase 1 – Being aware of the symmetrical pattern on the butterfly wings

At the beginning of the activity, the teacher told the story about the life circle of the butterfly. Then the teacher showed some kinds of butterflies' wings and asked the students to discuss about the special pattern on the butterfly wings. The students contributed by sharing their ideas and experiences. The activity then continued with constructing the butterfly wings. The students worked in pairs to put dots on the butterfly wings as many as the number demanded. The classroom mathematical practice that became establish as students participated in these activities involved the informal understanding about symmetrical pattern and representing a number in symmetrical pattern. Students came with the word "the same" and "similar". In this level, we do not distinguish the word "symmetrical", "the same" and "similar". In students' discussion, those terms to refer to symmetrical pattern.

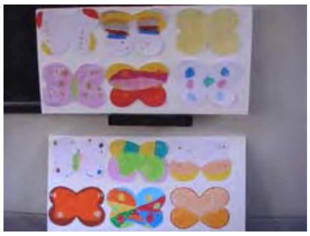
The emergence of the first mathematical practice – expressing a number using the symmetrical pattern

On this activity the teacher introduced the scenario of the butterfly wings. She started by discussing about the symmetrical pattern on the butterfly wings. Teacher (T) showed the picture of a butterfly wings. She would like to invite the students (S) to be aware that the wings of a butterfly are symmetrical.

- T: Do you see a kind of dots on this butterfly wings?
- S: Yes
- T: What are the colours of the dots in this side (pointed the left side)?
- S: White and Pink… Red
- T: Yes, white and red. How about this side (pointed the right side)? Do you also see the white and the red?
- S: Yes
- T: They are the same (colours), aren't they?
- S: Yes
- T: How many dots do you see in this side (pointed the left side)?
- S: Four
- T: No, only this side (pointed the left side)
- S: Two
- T: And this side (pointed the right side)?
- S: Two
- T: How many altogether?
- S: Four

From the dialogue above we can see that the role of the teacher was very dominant. She tried to come to the goal by guiding the students to answer her questions. We saw this as the traditional norm in that classroom, that the role of the teacher is giving the knowledge. Of course this was not the socio mathematical norm that we expected. In this episode, on answering the question, there was no need for the students to think about the questions. They could answer the questions just by looking at the pictures. There was no negotiation between the students about the accepted answer. Thus we made a note to this incident, that the activity did not give enough chance to the students to develop the meaning of the symmetrical pattern. In chapter 6, we would suggest activities as the refinement of this activity. From this episode we could not make a justification whether the students have already really become aware of symmetrical structure of the butterfly wings.

The lesson then continued by making their own butterfly wings. The teacher distributed the butterfly wings-shaped paper to the students. She asked the students to make their own butterfly wings. The pictures below are the result of students' work.

From those pictures we might see that students tried to make the butterfly wings in the symmetrical ways. The colours they chose and the patterns they drawn were more and less symmetrical. Although in the previous activity we could not see that students were aware with the symmetrical structure on the butterfly wings, in this activity we could state that students understood the symmetrical idea.

The activity then continued with constructing the structure by configuring the dots in the symmetrical pattern on the butterfly wings. The teacher grouped the students in couple. We expected that by asking the students to work on group, the mathematical practice would emerge during the interaction. Then the teacher distributed the models of a butterfly wings and the dots. One set for each group. She also distributed the working sheet for each child. She questioned the students to make a butterfly with certain dots and record their result by drawing it in the working sheet.

Observation on Iksan and Ari's works: Constructing five

In this episode the teacher asked the students to make a butterfly wings with five dots on it.

	Iksan	**Ari**	**Observer**
1	Put the first dot on the right upper corner. *Said: one* .He did not look at Ari.		
2	Put the second dot on the right below corner. *Said: two.* He did not look at what Ari did.	Put the dot on the butterfly wing by **mirroring** what Iksan did.	
3	Put the third dot between the first and the second dot in line. *Said: three.* He did not look at what Ari did.	Put the dot on the butterfly wing by **mirroring** what Iksan did.	
4	Put the fourth dot next to the first dot. *Said: Four.* He did not look at what Ari did.	Put the dot on the butterfly wing by **mirroring** what Iksan did.	
5	Put the fifth dot next to the second dot. *Said: five.* He did not look at what Ari did.	Put the dot on the butterfly wing by **mirroring** what Iksan did.	
6	Look at what Ari did.	Put the dot on the butterfly wing by **mirroring** what Iksan did.	
7	Looking ahead.	*Said: Finish… easy...*	
8	Counted the dots in their butterfly by tag each dot, started from Ari's side. Synchronized the tags with the sound. *Said: One, two, three, four, five, six, seven, eight,*	Looked at what Iksan did.	

	Iksan	Ari	Observer
	nine, ten.		
9	Gave signal with the hand that something was not right. *Said: No.*	*Asked the teacher: Wasn't it right? Five and five?*	
10	Took out the dots in Ari side while listened to Ari and looked at the observer.	Looked at the observer.	*Answered: what was the teacher said?*
11	Took out the dots in his side.	*Answer: Five*	*Said: Five, wasn't it?*
12	**Put one dot on the right upper corner.**	Looked at what Iksan did.	*Said: did it together.*
13	**Put one dot on the right bottom corner.**	***Said: two...* Take one dot.**	
14	**Put one dot on the left upper corner.**	**Put the dot on the left bottom side.**	
15	**Put one dot on the middle.**	**Pointing the middle.**	

Students change his work

In this observation we could see that Iksan represented the 5 by putting the five dots on the butterfly wings. The problem appeared when Iksan put the dots only in one side of the wings and Ari copied what Iksan did. She put the dots on the same arrangement as Iksan, on the other side of the wings. It seemed at that time they took their own portion. Iksan and Ari had their own side. Then Iksan reflected their work by recounted the dots and he found that the total is ten. He realized that there was a mistake there. He tried to rearrange the dots. Ari still did not see the problem, and she asked the observer: wasn't it right? In this happening Ari did not communicate her confusion with Iksan, instead she asked for scaffolding from the observer. The observer gave the support by reminding her to the instruction. Iksan then start putting the dots on the butterfly again, and Ari tried to participate on his work. Now they seemed to realize that the dots should be in the whole wings, not only on the one side.

In normative process, we conjectured that Ari's action asking for the support from the observer instead of communicating the problem with Iksan was influenced by the norm in that classroom, teacher decided whether an answer correct or not. We could also see that there was not negotiation between them to solve the problem. It seemed that they work individually, and each of them knew already what they had to do.

In mathematical process, the step when Ari mirroring what Iksan did showed that she understood that the pattern on the butterfly wings should be symmetrical. Thus, she tried to make it symmetrical by mirroring Iksan's work. And when we observed the step Iksan took after he realized that there was something wrong in their work, we could see that Iksan also understood the symmetrical idea. His third step putting the third dot on the upper left side showed that he mirrored the dot on the upper right side. Ari give him support by putting a dot on the bottom left side. She mirrored the dot on the bottom right side. In this step she also used her understanding of symmetrical pattern on the butterfly wings. The decision on putting the last dot in the middle was also evidence that they understood the symmetrical idea.

Students' Work: Configuring dots on the butterfly wings

In giving the instruction, the teacher started with the even numbers. As it has been explained in the HLT, this decision was taken as we predicted that the students would put the dots evenly on the left side and on the right side. Thus they would find difficulties on the odd number since they have to put the last dot in the middle. We conjectured that on doing that students need to think about the half.

In the actual learning process, our conjecture seemed not true. On putting the 8 and the 6, around 20% students has already put some of the dots in the middle (see attachment 2). This was a good incident during the learning process. The teacher then asked these students to show their work to the others. We expected that the other would able to take the knowledge from this sharing. In the later learning process students found no difficulties to solve 5 and 7. The teacher tried to dig up the students reasoning on putting dots on the middle.

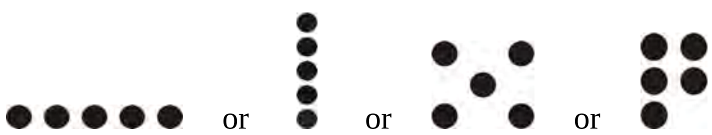
- T: How many (dots) in this side?
- S: Two
- T: And this side?
- S: Two
- T: I also see there is a dot in the middle. Why do you put it here?
- S1: To make it nice
- S2: **If you put it on the one side then it became unequal**

From that answer we see that the second student realized that they had to keep the wings symmetrical. He reasoned that putting the dot on the middle would keep the wings equal. Here he used the word equal to tell his understanding about symmetrical. It seemed that putting the dot on the middle to make them equal was a natural way. We conjecture that they have developed this knowledge from their common sense about the symmetrical pattern on the butterfly wings. Thus, our prediction about the students need to know the idea of half to solve this problem, was not completely true. Moreover, in other discussion, we may talk about using this common sense to develop the idea of half.

b. Phase 2 – Developing the framework about various representations of a number

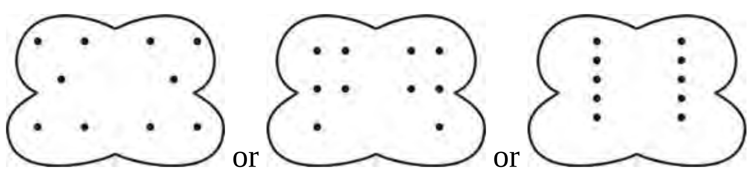
In the next activity, we prepared the magnetic board with two butterflies' wings on it, and also some magnetic buttons. The teacher said that in the previous activity she saw some children made different arrangement of dots on the butterfly wing for the same number. Then she asked two students to come in front of the class to show different dots configuration for a certain number. She also questioned the students to write the number above the wing represented the number of the dots in the left, in the middle, in the right, and also the total. The teacher explained that by writing the number above the wings they could categorize the sort of butterfly. We imagined that we would have benefits on writing this number. In the later lesson these numbers would be used to give insight to the children to see the relation between the numbers.

But as the students engage in this activity, we realized that our expectation did not meet with the students' thinking. We realized that the instruction about writing the numbers emerge different perception about structure. Putting this number meant we were to fast to move the students to seeing the configuration in the numerical pattern. These students in their level interpreted 'different structure' visually, in geometrical structure. Different for them meant different in the relation between the dots. Meanwhile, we defined two configuration were different when they resulted different numerical pattern or different number relation. We expected that they could see that a set of number is invariant under transformation. Thus, the configurations like:



for us, were expressing the same number.

Thus, for us,



were expressing the same (numerical) pattern 5 – 0 – 5. The students would think that those configuration were different as they saw the different position of the number.

Mathematically, we saw that our expectation was seeing the numerical pattern. And this expectation collided with students' view that seeing the configuration as the relation between the dots. We realized that we need to bridge this gap with some additional activity to shift the students' view on seeing the geometrical structure to seeing the number structure.

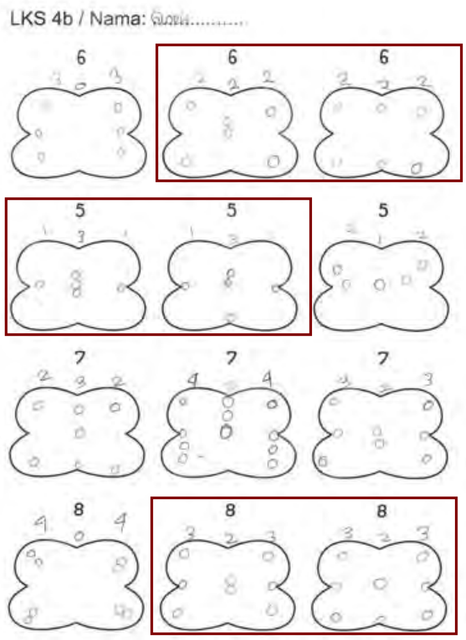
The emergence of the second mathematical practice – various configurations to represent a number

We prepared the magnetic board with two butterflies' wings on it, and also some magnetic buttons. The teacher asked two students to come in front of the class and show different arrangements for a number and compare the result.

Mini Lesson about various structures

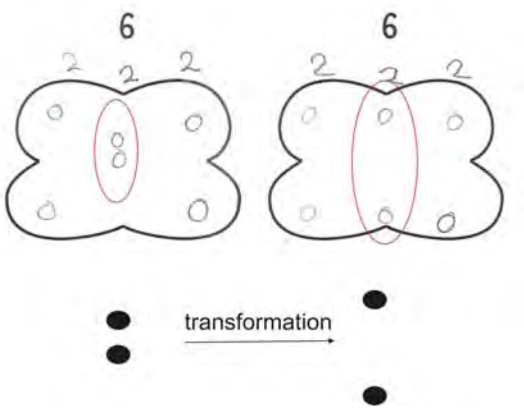
Observation on Gloria's work: expressing a number in various representation

In our observation we realized that the students catch the word of 'different' not in the same way as our expectation. We expected the different numerical pattern, whilst the students showed the different geometrical structure. We realized that our expectation in too high for this level of the students. But, observing the result in the students' point of view and back to our mathematical goal in this activity, we would say that during this activity, these students were able to express a number by showing different dots arrangements.



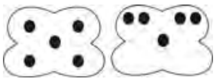
From the picture above, we could see how Gloria translated the instruction in her works. For her, on expressing the six, she gave three different structures. The middle dots in the second and the third picture, for her, were different in the position. Thus according to her, geometrically and visually, those two configurations were different.

For us, those two structures were not different. We referred to Freudenthal statement that said a set is invariant under a transformation.



Reflecting to this discord, we needed to think about the steps to bridge the gap. We expect that our next activity could help on bridging this gap.

c. Phase 3 – Grasping the relation among numbers by recording the possible configuration

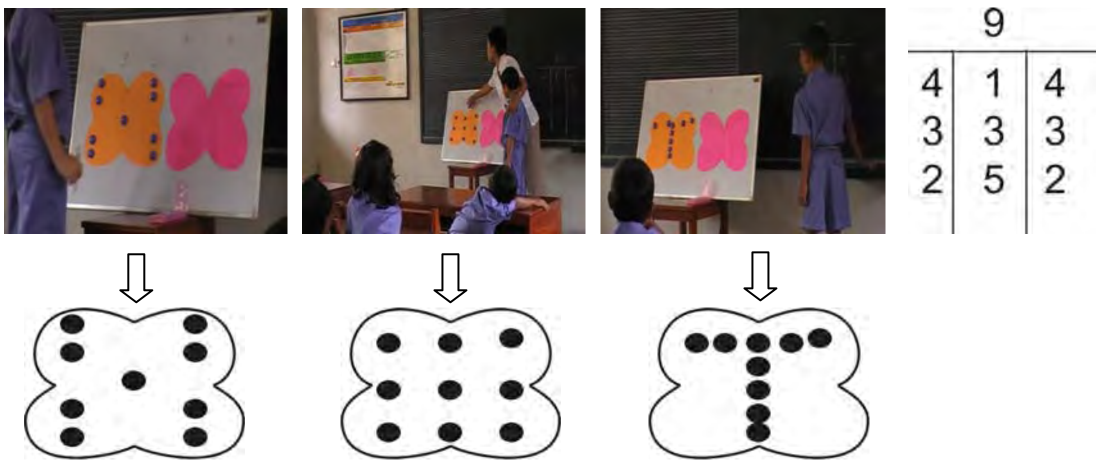
In this activity, the teacher introduced a table as a tool to support the student on recording the possible arrangement. The teacher told to the students that they need to register all kind of butterflies that they have. They need to classify the butterfly based on the same number they expressed but in different dots construction. When students discussed about the dilemma to decide whether  were different configuration or not, all of them still saw those arrangement visually. Thus, they judge that those were different configurations, even though when they put the number above the wings, they would get the same structure. They put those numbers also in the table although the numbers had already existed in the table. We reason that the students need to have more experience on the discussion of various arrangements the dots on the butterfly wings.

The emergence of the third mathematical practice – relation of number and each other

In this activity, a table was introduced as a tool to support the student on organizing the structures. Using the table meant the students had to go to higher level mathematics. They should leave the level where they always worked with the model of butterfly wings and the pictures and shift on working with number.

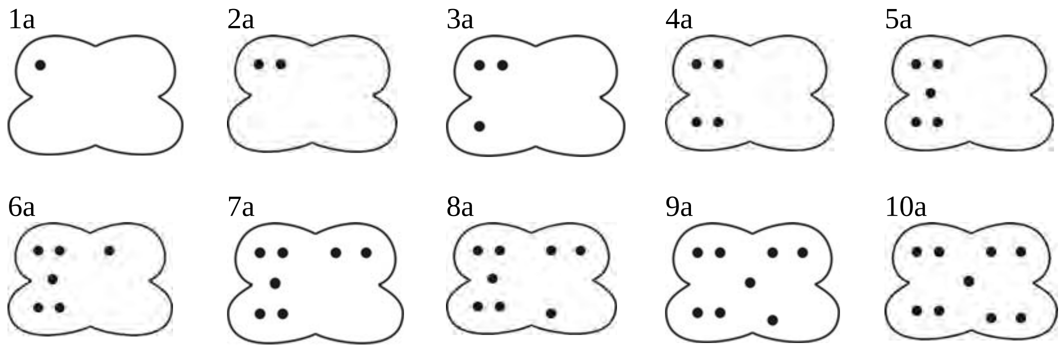
Observation on the mini lesson: tabling the structure

The teacher asked the students to come in front of the class to show the construction of 9. Before, she prepared the table with three columns on the blackboard.

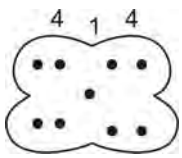


First the students should propose the arrangement that were different from others'. Then the students should write the numbers above the wings, and then moved the numbers in the table on the blackboard.

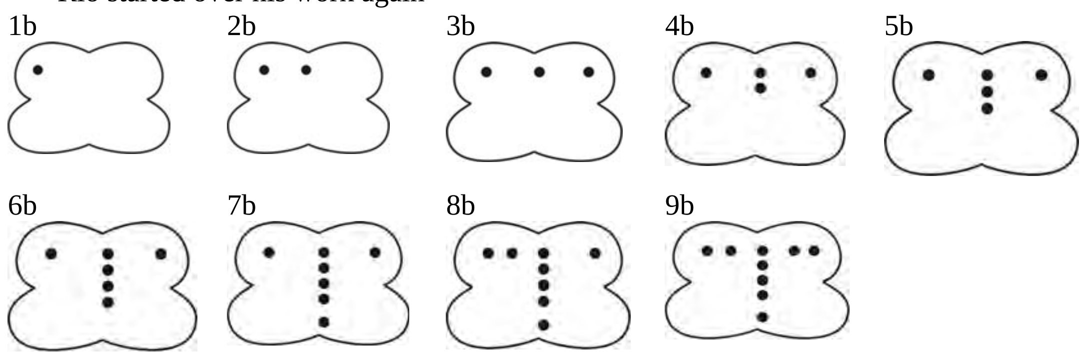
As it was predicted, the students faced the same problem as before. The third child in the picture above (Rio) came in front of the class. First, Rio tried to make the arrangement of 9 dots as below.



- The teacher then asked him to write the number above the wings



- **The teacher questioned the class whether the configuration was right**
- The class agreed that that was right as the left and the right side are equal
- The teacher questioned whether the cobfiguration was the same as the first boy
- The students said that it was different.
- The teacher asked the Rio to look at the table
- The teacher questioned whether Rio saw the same numbers on the table (referring to the first row of the table)
- Rio was nodding
- **The teacher said that Rio did not do something wrong. But the teacher want another configuration that could give different number on the table**
- Rio started over his work again



. The reason he shift the dots on 8a to position on 9a was Rio realized that he only had one dot left. And when he put the last one on the right bottom side then the wings became unequal. Then Rio thought that he needed to shift the dot to the middle to make the wings symmetrical. This showed that Rio had already applying the symmetrical structure on his work. When he started putting the dots first on the left side of the wings then moved to the right side, showed that he grouped the dots in two set, left and right. Even though, later he also needed to think about the middle. In his second trying, he set already three groups: left, middle, and right. Then he started to fill the middle, and add one more on the left and on the right.

In the normative observation, we saw that the role of the teacher started to change compared with her role in the first lesson. She gave more chance to her students to decide whether the answer was correct or not. And by saying that Rio did not do something wrong, but she wanted another configuration that could give different number on the table, showed that the teacher established the norm on accepting other answer, and asking for further explanation.

Refers to the aim of this lesson, the table are able to show them the relation of number to each other. The students may see that a number has various kinds of relation. By using this table, we expected that students have different images of a number that is a part of number sense. In their later learning, by knowing various relations of numbers to each other, the students will be flexible to handle the operation of numbers. For example, if in the later lesson they have to solve 9 + 6, they knew that 9 can be 4 + 1 + 4. Thus, 9 + 6 = 4 + 1 + 4 + 6 = 5 + 10 = 15. And to solve 9 + 8, they will choose 9 as 2 + 5 + 2, because 2 and 8 make 10.

But it ought to be admitted that moving from the model of the butterfly wings to the table was not an easy step for the students. We observed that students would able to do the task fluently when they have more time to do the discussion.

d. Phase 4 – Developing fundamental methods for arithmetical reasoning by double and almost double

In these activities, the teacher started the scenario with telling the story about butterflies who were perching on the branches. Then she asked the students to determine the number of the dots on the butterfly wings. There were three tasks in this phase. First, students needed to find the pair of the butterflies' wings. Second, they were given pictures of half butterfly wings. Then the teacher asked them to draw the other half of the butterfly wing. Third, the teacher distributed the working sheet with the pictures of butterflies on it. Since the students could not see the wings of the butterflies completely, then they had to find the strategies to determine the number of the dots. On the first and the second task most of the students perform well. They used their knowledge about the symmetrical pattern in the butterfly wings to do the task. Nevertheless, we made a highlight on the process of this accomplishment. Since in some task we used half dots, some students might have seen these in the different way. They did not recognize the half, thus they either leave the half as it was a whole dot or mirrored the half since it was a whole dot.

In the third activity, as the mathematic level of the students was not yet high enough, our intention to emerge the double seemed not well accepted. Instead of doubling or almost doubling, the students used one – by – one counting or counting on strategy to solve the problem. We conjectured that the experience that students need to come to this strategy was not enough. Thus, we would propose some additional activities or discussion to come to this goal.

The emergence of the fourth mathematical practice – double and almost double

The third activity for this phase was determining the number of dots on the butterflies' wing. The teacher distributed the working sheet with the pictures of butterflies on it. Since the students could not see the wings of the butterflies completely, they had to find the strategies to determine the number of the dots. On having those strategies, first the students had to realize that the wings were symmetrical. Thus they have to double the amount of dots that they saw.

From the observation, there were three happening that were interesting. The first one, on solving the problem, the students counted the same dots twice. The second one, the students counted the dots then flipped around the paper and they counted the shadow of the dots in the back side of the paper. The third one, the students did the counting on. They recognize the amount of the dots that appeared then pointing the dots while saying the next number.

From the 17 students, 2 gave all the answer correct and 10 students made 1 mistake. This common mistake was from the problem included the half dot.

There were 15 of 17 students who counted twice the half as the whole dot. Thus they had 6 instead of 5. Two students could see it as half, thus they knew that the dot in the middle is only one. Reflect this result to the result that students did in the lesson of pairing the butterfly wings, it came to contrast. The students were able to pair the half to get the whole dot in that lesson. It seemed that the imagery from that previous learning was not caught well by the students. They understood the symmetrical idea, they started to understand the double idea, but they did not recognize the half.

Our reflection was this instruction was not really appropriate to evoke the almost double idea. From the previous learning experience, students used to work with three set of dots: the left wing, the middle, and the right wing. In this instruction, unintentionally we lead the students to see the wings in two set of dots and we occupied half to eliminate the third set. But we thought that this instruction was proper to evoke the double concerning the even number.

e. Phase 5 – Advancing the counting strategy

In this phase, the interest of the children moved from working with single butterfly to working with the group of butterflies (flutter). The teacher started the activity by telling the story to engage students' awareness about the pattern on the butterfly wings and their common sense from their daily life observation that butterflies fly in a group. The teacher questioned them to determine the total amount of all dots in the wings. Some students still do the one – by – one counting. But some of them started to recognize that the dots on the group of the butterflies were the same for each butterfly, and by a support from the teacher they started to develop skip counting strategy.

Then activity continued by making a flutter which total amount of their dots represent a certain number. The students engaged in this activity in two ways. The first group of the students employ their experience from the previous activity. Thus, they chose the wings which had the same amount of dots to compose the required numbers. The second group of the students chose any kind of wings to fulfil the number. The observable incident showed that both group of students counted the total dots one by one to give the result. But when we observed further in the work of the first group of students, their decision on choosing the wing in pattern showed higher mathematical reasoning. They started to develop the numerical pattern, instead of only geometrical pattern.

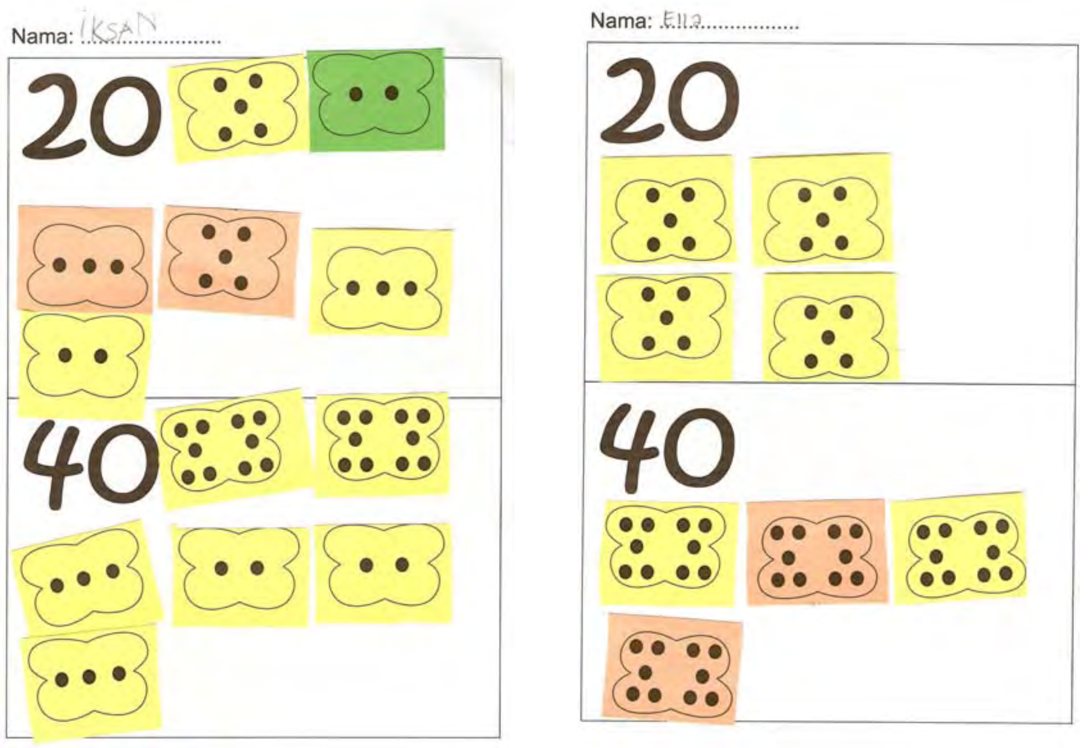
The last activity in this sequence was similar with the previous activity. But we replace the dotted – butterfly wings with numbered – butterfly wings. Observing on students works, they start to develop the skip counting strategy. They could perform well on the task of skip counting by two and by ten.

The emergence of the fifth mathematical practice – skip counting strategy

The lesson continued with mini lesson about the group of the butterfly who has the same amount of dots in their wings. The children were asked to determine the total number of all dots in the wings. The strategy that was trying to develop was the skip counting. It was expected that the students could move their strategy from one-by-one counting to skip counting by using the structure of the dots in the butterfly. It was an introduction activity to lead the students to come to the main activity.

The teacher distributed the working sheet with numbers and set of butterfly wings with various kinds of dots. They were asked to make a group of butterfly in such a way that the total dots of that group fulfilled the number that was questioned.

Comparing Iksan's and Ella's work: choosing the wings



From the work of Iksan we saw that the wings he took had various dots. It seemed that on doing the tasks Iksan did not obey the pattern. His aim was one: fulfilled the number. Thus, it was predicted that Iksan chose the wings by counting the dots one by one so that he perform the task. From the work of Ella, we can see something different. Ella looked more patterned on choosing the wings. On accomplishing the task she used the same group of dots for each wing. It was predicted that Ella recognized the group of the dots and recognize the pattern of the number. Even though on doing some other problem she counted the dots one by one, the wings that she picked up were not random. Her choosing indicated that she started to employ pattern in her counting, although she did not completely move to skip counting strategy.

Observation on Ari's work: fulfilling the 20

First, Ari took some wings with five dots. Then she took one wing and counted the dots: *one, two, three, four, five.* She took the second wing, counted the dots: *six, seven, eight, nine, ten.* She took the third wing, counted the dots: *eleven, twelve, thirteen, fourteen, fifteen.* Then she took the fourth wing, counted the dots: *sixteen, seventeen, eighteen, nineteen, twenty*. Then she moved away the rest of the wing. After that she started gluing the wings she collected.

This is an example where the student got the idea of patterning. She might not be able to do the skip counting yet, thus to be sure, she counted the dots one by one. It will look different with the students who chose any random wing and counted the dots. Sometimes it would result that many time he has to change the choice as it did not fit with the task. We can see that in the following observation.

Observation on Tika's work: fulfilling the 8

Tika had already stick the three wings with three dots on the paper to make 8.

- Teacher (T): Are they 8?
- Tika (Ti): Yes
- T: Really? How many are these? (pointing dots on the first wing)
- Ti: Three.
- T: And these? (pointing dots on the first and the second wing)
- Ti: eemm..
- T: Let's count. (tagging each dot)
- Ti: one, two, three, four, five, six, seven, eight, nine
- T: You are asked to make eight.
- Ti: (Took out the last wing. Took another wing with three dots on it. Counted the dot. Took another wing with two dots. Counted it again)

It seemed that Tika did not use pattern and she did trial and error to fulfill the 8. She has not had any idea of this learning. For her, the dots are the dots. They are separated object that are not patterned and not grouped.

In the next activity, we chose only five students to be experimented. We realized that this activity was very formal. And reflecting to the students' situation we thought that not all students ready to do this activity.

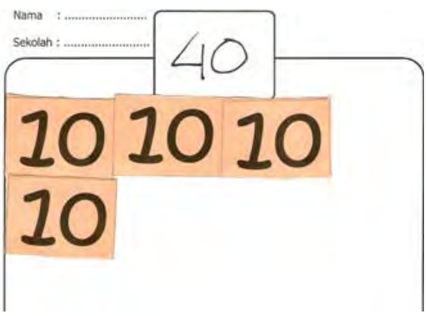
The teacher distributed the two sided butterfly wings. We put dots on one side and number representing the dots on the other side. It was aimed that students could still see the dots if it is necessary for them.

- Teacher (T): Put a wing with the 3 on it and questioned the students: *How many dots do we have?*
- Student (S): *Three*
- T: Put another wing with three next to it. *How about this?*
- S: *Six*
- T: Put another wing with the 3
- S: Silent. Counting with their hand. Said: *Nine*

In this phase we knew these students still needed to concretize object to support them. After some other conversation the lesson continued. We distributed a working sheet with number on it and also the wings with numbers. We did not use the dots anymore. We asked the students to make a group of butterfly in such a way that the sum of the number they choose fulfilled the number that was questioned. We also expected that they give more than one kind of answer. With these students, our expectation could meet their thinking process. They could employ the numerical pattern and started to develop the strategy of skip counting. One interesting incident, when Wihas has to solve the problem related with forty. His first comment when he saw the problem was: "It is easy"

- Teacher (T): *why is it easy?*
- Wihas (W): *It is ten.. and ten.. ten.. and ten..*
- T: *and…?* (tried to conflict his mind)
- W: ... (did not give any answer then stuck four tens on the paper)

Wihas did not saying another ten, because here he realized that forty was only four times tens. Here we saw that Wihas reduce his counting process using the skip counting strategy.



Reflecting in our observation on students' work, we conjectured that two, five, and tens were easy to evoke the skip counting strategy.

5.4.3. Final test to find the evidence of the learning process

We interviewed ten students to find the evidences of their learning process. We arranged some short task interviews that were done individually. In this task interviews, we encourage them to show their ability to represent number in a pattern. We also wanted to see their counting ability, whether they employed pattern to help them ease the counting process. We gave loosed – buttons and some boxes. Those boxes could contain ten buttons in each box.

First, we asked the students to tell how many buttons they had. From the strategy they performed, we tried to see whether they used structure to support their counting process. Then we introduced the boxes to the students. We asked them to put some buttons in the boxes to show that there were maximum ten buttons in each box. We questioned them to show, for example sixteen buttons. We would see if they used arrange the dots to help them showing this amount.

From our observation in the students' strategy on counting process, we saw three differences. Two students counted the pile of the buttons by tagging and not moving the buttons. It resulted that they recounted the buttons that they had already counted more than once. These students did not organize the objects that were being counted. It seemed for us, they did not do structuring during their counting process. Some other students tagged the objects one – by – one and shift them to the side and arrange the object that had been counted in structured-like arrangement. We said it 'structured-like' as it was not really a structure. We saw that these students start to organize their counting process although they have not done it in a very proper manner. The rest of the students were counted the object by skip counting by two. In our observation we guessed sometimes on doing this they still counted by one in their heart. Nevertheless we saw that these students tried to ease their counting process and advance their strategy. They reasoned that by counting in that manner they shorten the counting process.

On the second task, when we asked the students to show a number using the representation of button and boxes, we observed that they did not perform well. We realized that our choice of giving this task was not appropriate and not relevant with their learning process. In this task they had to recognize the structure of the box and the buttons in a very short time. We became conscious that structure recognition was not an easy process for these students. Thus, we could not find the evidence of their learning process from this task. It would have been better when we used structures that had been known by these students.

5.5. Conclusion

In this section I will present my conclusion about this cyclic development by answering my research questions.

a. What are the roles of patterns to support children perceiving numbers?

The symmetrical pattern on the butterfly wings could support the students on perceiving the number by having the representation of number and knowing the relation of number to each other. In the actual learning process I observed that there are three kinds of students' work during the teaching experiment: (1) students pattern the dots – students repeated the configuration on the one side; (2) students structure the dots – they made configuration in such a way that they have recognize (e.g. Iksan and Ari's work on making the 5); (3) students organize the dots – students put the dots in a symmetrical configuration, but they did not either pattern the dots or configuring it in a structure. Those three kinds of students work will give the imagery to the students about the relation of number to each other and also the number representation. But pattern will give more support on the development of the later idea – like double and later strategy – like skip counting.

b. How can children develop their counting strategies by patterning?

One of my proposal in this research is the students will develop their counting strategy by patterning. First, they will develop the idea of double (and almost double). The students have a configuration of dots in one side of the butterfly wings. Then they repeat this configuration on the other side. They made a pattern, which in this activity is symmetrical pattern. By doing this they will see that they have twice (or almost twice) more dots than the initial dots that they have. This experience may give imagery to the students when they do more exploration about the idea of double and almost double. Second, they will develop their counting strategy into skip counting. In this development, students need to shift their understanding of pattern as the spatial configuration and the relation among the element to understanding pattern in a numerical repetition.

c. *What is the role of socio-mathematical norms in motivating children on perceiving number and developing counting strategies by patterning?*

The ideal condition is socio-mathematical norms will give contribution to an individual learning process when a student construct their knowledge by the taken – as – shared practice. This will happen when the students share their knowledge and negotiate and reason the acceptable solution. When the students proposed their idea of configuring the dots to represent a number, the students saw that there were more than one way to represent a certain number and there were many kind of relation of number in a certain number. Students had developed their counting strategy to skip counting when they share the idea during the class discussion on how to shorten the counting process. They reason using the (numerical) pattern they could ease the counting process.

d. *What kind of contextual situation and means that support children to perceive numbers and develop counting strategy by patterning?*

In this sequence of activity, there were three contextual situations that were mathematized to support children to perceive numbers and develop the counting strategy by patterning: (1) the symmetrical pattern in the butterfly wings – using this symmetrical pattern, students represented the number by configuring the dots on the butterfly wings; (2) the perching butterfly – using the fact that the butterfly wings are always symmetrical, the students could determine the amount of dots on the butterfly wings although they only knew the amount in one side; and (3) the flutter – by seeing the broader situation (shift from one butterfly to the group of butterflies) the students could reason using pattern to shorten their counting process. A table to record the possible configuration of dots on the butterfly wings give support to the students to see the relation of numbers to each other.

6. Discussion

Doubt is the father of invention
(Galileo Galilei)

I will present this chapter in two sections. The first section is about the reflection of this research and the reflection about my learning process during doing this activity. The second section is about my recommendations to improve the sequence for the next cyclic of the development.

6.1.Reflection

I designed the sequence of activity in this research with the intention using the realistic approach. But in the practice, I observed that I did not give enough support to the students to develop their own model using their internal representation. The sequence of the activity was too dense and too fast – too much mathematical concept that they have to achieve only in 10 days of learning. It caused the teacher took a big role during the learning process by determining the acceptable solution and giving the model that had to be used. The students had a very small chance to develop their own learning by discussing and negotiating, instead they needed to take over the meaning from the teacher. It seemed that I used the contextual situation only as the motivation of the learning process, not as the situation that to be mathematized by the students. I needed to really learn about the students' point of view by conducting the proper pre-test, so that my design will be appropriate to support children's development.

Having conducted this research, I realized the importance of contextual situation, models, and the influence of socio-mathematical norms to contribute students' learning process. Otherwise, we will back to the old paradigm – where the teacher transfers the knowledge to the students.

Activity 1: Observing the butterfly wings – Is that enough?

In this activity, students were asked to observe and to compare the pattern on the butterfly wings and find the resemblance to each other. We expected by doing this activity students would come to the awareness about the symmetrical pattern on the butterfly wings – which in the subsequent activity this awareness would be used to build the representation of numbers. I observed the discussion and the learning process. I see that by doing this activity, the children were only accepting the expected solution.

Activity 3: Put the number above the wings – Is it too early?

In this activity students were asked to put the number above the wings to represent the amount of the dots in the left side, the middle, the right side and the total. Then the students had to present different dots configuration. I observed that it caused collision on the students meaning about different configuration. In their level, because they work in concrete level with dots, configuration related to the relation of the position of dots to each other. In my expectation, the different meant that the structure produced different numerical pattern.

6.2.Recommendation

Based on my reflection, I will suggest some recommendation for the next cyclic of the development.

Activity 1: Making my own butterfly wings

I propose another activity using paper and paint to make children's own butterfly. The goal of this activity is: students will experientially real about the symmetrical pattern on the butterfly wings. I propose the intended activity as the following. First the students explore the happening when they put the paint in one side and see what happen after they folded the paper (the students will see the mirror image of the paint in the other side). Second, the students can make a certain pattern on the one side of the paper and then reason and predict about the image of a certain pattern. I envisage that by doing this activity, students can have their own model about the symmetrical pattern on the butterfly wings.

Activity 3: Not necessary putting the number now

To refine this activity, I propose that in this activity the students do not need to put number above the wings. Instead, I give them as many as opportunity to make all possible configuration and record the result by drawing (not number). By doing this, I give chance to the students do more exploration on dots configurations. To bridge the movement to a number, I propose to make another activity, where the dots on the wings are replaced by number.

Socio-mathematical norm

The emergence of expected socio-mathematical norms that can give contribution to the students on developing their number sense is strong related to the instructions that were given. We need to see the condition of the students so that we can develop activities to surface the expected norm.