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Structures Supporting the Abbreviation of Addition Strategies up to 20

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Abstract

The goal of this research was to develop theory and classroom activities that support children's learning on addition up to 20 by using structures. This research sought to answer of the question, how can structures support students' learning of abbreviated strategies of addition up to 20? A design research was conducted in which the theory of Realistic Mathematics Education underpinned both the students' learning process and the design of the classroom activities that support that learning. Special attention was given to students' learning trajectory that aimed at promoting students' awareness of structures and students' own constructions, at employing structures to move from counting all to counting by grouping, and finally at using the structures for abbreviating strategies of addition up to 20. The results of this research show that the structures support students' development of addition strategies; from counting strategies to more abbreviated strategies.

Key words: structure, addition, double strategy, decomposition to 10 strategy, design research

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Chapter 1

Introduction

An enormous number of studies has been conducted to investigate how children learn addition strategies. Villete (2002) found out that children do not reason arithmetically when solving addition and subtraction problems. Their ability to add and to subtract is based on physical representations and the use of one-to-one correspondence. In Nigeria, Adetula (1996) revealed that children who learned addition and subtraction in a traditional-method classroom environment, in which addition and subtraction were introduced in terms of joining and separating sets using concrete objects before students are drilled on abstract problems, resulted in a lower achievement than those who learned in a constructive classroom environment.

In understanding the addition concept, Torbeyns, Verschaffel, and Ghesquière (2004) investigated the differences in strategy characteristics and development between children with high mathematics achievement and children with low mathematics achievement. She revealed that high mathematical achievers use retrieval more frequently than counting. This strategy implies a better and more accurate answer than relying on counting which produced more errors.

Canobi, Reeve and Pattison (2003) showed that children's development differs on a wide range, from counting all to mental arithmetic. Children who understand the way numbers are decomposed and combined used sophisticated strategies of problem solving. The teaching of arithmetic at school should foster students with low understanding of number relations so that they can gradually transform their informal counting strategy into a more efficient and flexible strategy.

The current condition of mathematics education in Indonesia has been reported by Sembiring, Hadi and Dolk (2008) which showed the problem in primary education that students have difficulties to comprehend mathematical concepts, to construct and solve mathematical representations from contextual problems. This problem is caused by the traditional teaching-learning method where the teacher is the central person and the knowledge is transferred by telling. In this method, students learn standard algorithm as a fixed procedure of solving problems. Armanto (2002) revealed several misconceptions that resulted after students learned a standard algorithm. Some teachers argued that by learning a standard algorithm, students can apply it to solve problems easily. This indicates the teacher's belief of teaching mathematics, that mathematics is a set of standard fixed procedures. This belief causes obstacles for students to mathematize therefore mathematics becomes meaningless. As a result, there is a gap between school-taught mathematics and students' actual mathematical abilities.

More specifically, in learning additions, there is a gap between school taught-formal algorithm and students' actual ability. In school, the teacher teaches the students a formal procedure of addition. When solving addition up to 20, students wrote a formal notation of adding two digit numbers where the tens and the one are separated. This notation is not appropriate to solve additions up to 20 problems because additions up to 20 do not have tens and ones. For example, to solve 8 + 5, students' wrote a formal notation where the numbers are ordered in a column.

$$\begin{array}{r} 8 \\ \underline{5}\,+ \\ 13 \end{array}$$

Figure 1.1: Students' formal notation

This notation does not support students' thinking of solving the problem. Consequently, students do not acquaint to other strategies but a primitive strategy such as counting. This indicates that students do not have a meaningful understanding about the algorithm and their mathematical knowledge is not based on their common sense.

There is a gap between students' actual performance and teacher's expectation. It is a challenge to improve the mathematics teaching and learning in Indonesian schools. Realistic mathematics education offers an opportunity to change mathematics education in Indonesia by giving students the opportunity to mathematize. In realistic mathematics education, students are the active participants in the classroom where they construct their own understanding.

The goal of this research is to develop classroom activities that support students' learning of abbreviated strategies of addition up to 20 by using structures and add to local instruction theory of addition up to 20. We pose a research question, "how can structures support students' learning of abbreviated strategies of addition up to 20?"

Chapter 2

Theoretical Framework

2.1. Realistic Mathematics Education

The central principle of Realistic Mathematics Education (RME) is that mathematics should be meaningful for students, that students can experience mathematics when they are solving a meaningful problem (Freudenthal, 1991). His idea was “mathematics as a human activity”. Realistic Mathematics Education gives many opportunities for students to think and construct their own understanding. In Realistic Mathematics Education, pupils are challenged to develop their own strategies for solving problems, and to discuss those strategies with other pupils.

Treffers (1987) described five tenets of realistic mathematics education which are: (1) the use of contexts, (2) the emergence of models, (3) students’ own constructions and productions, (4) interactive instruction, and (5) the intertwining of learning strands. The first tenet, using contextual problems might stimulate students to think of ways to solve the problems. In RME, the point of departure is that context problems can function as anchoring points for the reinvention of mathematics by students themselves (Gravemeijer & Doorman, 1999). A rich and meaningful context is essential to begin the classroom activity.

A good context should allow students to mathematize, for example by using representations and models (English, 2006). Gravemeijer (1999) explained that the contextual situation serves as the starting point of students’ to conceptualize a more formal mathematics by modeling the problems. In an educational perspective, modeling requires the students not just to produce or to use models but also to judge the adequacy of those models (Doorman, 2005). In a classroom activity setting, Gravemeijer (1999) explained how a model can serve as *model of* a situation and transforms into *model for* a more formal mathematical reasoning. The level of emergent modeling is described in figure 2.1.

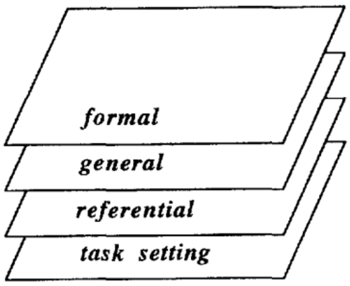


Figure 2.1: Level of activity in emergent modeling

1. Task setting level: in which the activity involves interpretation and solutions that depend on understanding how to act in the problem setting (often out-of-school setting.
2. Referential activity involves models that refer to activity in the setting described in instructional activities.
3. General activity involves model for that facilitate a focus on interpretations and solutions independent of situation-specific imagery.
4. Formal activity is no longer dependent on the support of models for to achieve mathematical activity. (Gravemeijer, Cobb, Bower, & Whitenack, 2000)

The third tenet highlights students' own constructions and productions where Treffers (1987) believed that students' construction stresses on the *action* of the students while students' production stresses on the *reflection* of teacher's didactical activity. The relationship between students' own constructions and productions therefore is not dissociable from the teacher's role. In a realistic mathematics classroom, students construct their own knowledge, guided by the teacher (Treffers, 1987). Freudethal (1991) used the term *guided reinvention* to name such students' construction. In his view, students should reinvent mathematics themselves by repeating the learning process of how the mathematics was invented. Students should experience the learning of mathematics as a process similar to the process by which mathematics was invented. However, with the guidance from the teacher, the process of reinventing mathematics can be made shorter than how it was invented in the history. The guidance can be given in activities that allow and motivate students to construct their own solution procedures.

The fourth tenet emphasizes the interactive classroom environment which promotes classroom discussions as a way of sharing knowledge among the students. An effective classroom discussion should lead students to express their ideas and solutions of the given problems, and at the same time to respond to each other's solutions (Cobb & Yackel, 1996). In such situation, students will be able to negotiate to one another in an attempt to make sense of other's explanation, to justify solutions, and to search for alternatives in a situation in which a conflict in interpretations or solutions have become apparent (Cobb & Yackel, 1996). A discussion should also center on the correctness, adequacy and efficiency of the solution procedures and the interpretation of the problem situation (Gravemeijer, 1994).

The intertwining principle of realistic mathematics education is often called the holistic approach, which incorporates application, and implies that learning strands should not be dealt with as separate and distinct entities (Zulkardi, 2002). With this approach learning mathematics can be more effective, for example learning algebra and geometry can be done simultaneously.

2.2. Early arithmetic: From counting to adding

Counting is probably the most natural way of determining the quantity of a collection of objects. Freudenthal (1991) believed that counting is a child's first verbalized mathematics. Gelman and Gelistel (1978) argued that there are three principles children need to learn to count properly. The first principle is one-to-one correspondence which obliges children to count objects only once, or otherwise they will get a wrong total. The second principle is about constant order which is closely related to the ordinal aspect. And the last principle is finding the total amount of objects being counted which is indicated by the last number mentioned in the counting. This principle stresses on the cardinal aspect.

Subitizing is a perceptual process of determining accurately how many objects are contained in a small set of objects (less than or equal to four and five) (Klein & Starkey, 1988). There are two types of subitizing; perceptual subitizing and conceptual subitizing (Clements, 1999; Charlesworth, 2005). Perceptual subitizing is instantly knowing how many objects there are without needing a mathematical process. Young children are usually able to do perceptual subitizing up to 4 items. On the other hand, conceptual subitizing is an advanced subitizing that requires more than just recognizing a quantity. Conceptual subitizing obliges one to recognize the number patterns as composite of parts and as a whole. For example, people can tell an eight-dot domino immediately by conceptual subitizing. They see each side of the domino as composed of 4 individual dots and as one group of 4 while the whole domino as composed of two groups of 4 and one group of 8. From the example, it implies that conceptual subitizing involves structuring (i.e., viewing pattern and regularity in the configuration of the dots).

Even though many experts are still investigating the relationship between counting and subitizing, some suggestions have been made. Benoit, Lehalle, and Jouen (2004) suggested that subitizing is a necessary skill to understand counting. On the other hand, Clements (1999) recommended that counting and subitizing can support one another. Young children use perceptual sibitizing to make units for counting and also use counting to construct conceptual subitizing.

An understanding that a collection of items can be made larger by adding is a fundamental aspect in everyday life which implies that addition is an important topic in early arithmetic (Baroody, 2004; Starkey, 1992; Geary, 1994). Nunes and Bryant (1996) explained that at early age, young children can perform addition when it can be modeled with concrete objects. At this age, children's addition is performed by counting (Kilpatrick, 2001; Ginsburg, 1977).When children enter school, their abilities are quickly expanded, instead of using concrete objects, now they can use number words to represent the addends.

2.3. Strategies of addition

Many experts have investigated children's strategies of one digit additions (Nwabueze, 2001; Torbeyns, 2004) and they revealed that children start learning about addition through counting and go through a long process before they reach abbreviated strategies. Moreover, Kilpatrick, Swafford, and Findell (2001) explained that children learn addition through a long progressive process in which they develop different strategies. First, children count all objects (counting all) and this strategy becomes abbreviated as children become more experienced with it. They don't need to count all objects but start with one addend and count on (counting on). In time children can memorize some sums and are able to recompose a number (e.g., 7 = 6 + 1). As they develop this skill, they begin to learn a more sophisticated strategy which is derived from the composed number such as a double (e.g., 6 + 7 = 6 + 6 + 1 = 12 + 2 = 13). This process is shown in figure 2.2

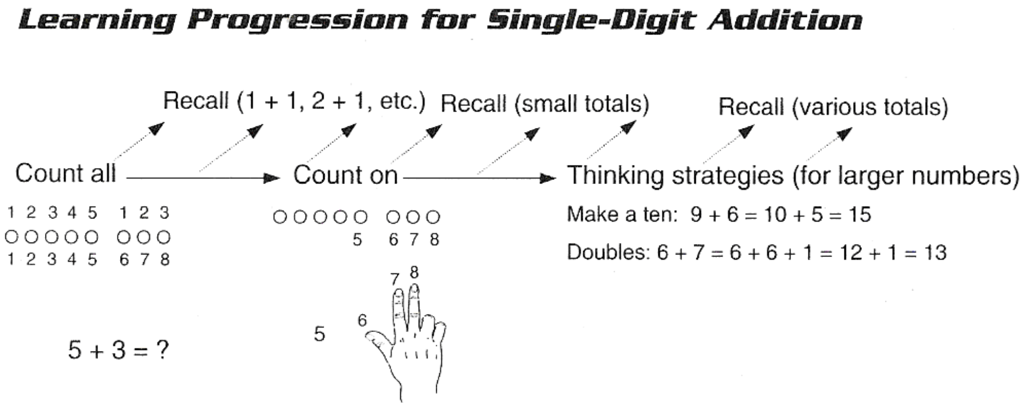


Figure 2.2: Learning progression for single-digit addition
(Kilpatrick, et al., 2001, p.187)

Torbeyns, Verschaffel, and Ghesquière (2004) added that the counting strategies are very likely done by using fingers. Fingers like other objects can be used to represent a number, and they can help children keep track of their counting (Geary, 1994). For children, this is a simple and natural way of determining quantity, thus if not stimulated to develop other strategies, they tend to keep on using the counting strategy. Reformers of mathematics education have suggested changing school mathematics in such a way that students are fostered to develop effective, flexible and meaningful strategies.

Strategies using decomposition of numbers allow children to use many combinations of number flexibly when they do addition. Such strategy are the 'tie strategy' (Torbeyns, et al, 2005) or 'double strategy' (Van Eerde, 1996; Kilpatrick, et al., 2001) that is 6 + 7 = 6 + 6 + 1 = 12 + 1 and the 'decomposition to 10' (Torbeyns, et al., 2005) or the 'make a ten'(Van Eerde, 1996; Kilpatrick, et al., 2001) where 6 + 7 is solved by 6 + 4 + 3 = 10 + 3 = 13. By learning these strategies children develop the flexibility to use the strategy that is efficient for them. For this research we use the term double strategy and decomposition to 10 strategy.

According to Torbeyns, Verschaffel, and Ghesquière (2005) the decomposition to 10 strategy consists of 2 steps. First, a child needs to decompose one addend into 2 parts in such a way that when one part of it is added to the other addend, it will make 10. Next, he/she has to add the remaining part to the 10. (e.g., 8 + 5 = … ; 8 + 2 = 10; 5 = 2 + 3; so 8 + 5 = 8 + 2 + 3 = 10 + 3 = 13). Van Eerde (1996) called these steps the 'building blocks' of a strategy. To be able to solve 8 + 5 by decomposition to 10, a child needs to know three building blocks, namely 8 + 2 =

10, 5 = 2 + 3, and 10 + 3 = 13. First, the child needs to find which number, when added to 8, will give 10 as the result. After he/she finds that number, which is 2, then he/she decomposes the 5 into 2 + 3. Finally he/she adds 10 + 3 = 13. Furthermore, Van Eerde (1996) specified one of the building blocks as the ‘friends of 10’ that is the addition that makes 10. The friends of 10 consist of a pair of numbers that makes 10 when added (e.g., 1 and 9, 2 and 8, 3 and 7, etc).

For this research, we defined more detailed steps of doing decomposition to 10, and we illustrate the steps with an example (i.e., 8 + 5) as follows:

1. Determining the starting point. Students can choose to go from 8 or from 5. For instance, if a child chooses 8, then he/she can go on the next step
2. Finding the friends of 10. In this step students find the friend of 8, which is 2.
3. Decomposing the other addend. Students will perform splitting the 5 into 2 and 3. 5 = 2 + 3
4. Adding 10 and the remained number from the splitting together. 8 + 5 = 8 + 2 + 3 = 10 + 3 = 13.

2.4. Relating structure and strategies of solving addition up to 20 problems

Freudenthal (1991) believed that structuring is a means of organizing physical and mathematical phenomena, and even mathematics as a whole. In term of physical objects, Batista (1999) described spatial structuring as mental operations of constructing an organization or form for an object or set of objects. Van Nes and De Lange (2007) defined spatial structure as a configuration of objects in space which relates to spatial regularity or pattern, for example, the configuration of dots in a dice. Structuring in this study is the operation of breaking and grouping objects as an attempt to organize those objects in a regular configuration.

Clemment (1999) suggested that structures can be used to foster students’ ability of counting and arithmetic through conceptual subitizing. Benoit, Lehalle, and Jouen (2004) found that when large numbers are hard to subitize, children rely on the presentation where they look for pattern in a configuration. Moreover, Stefffe and Cobb (1988) recommended using structures to help children develop abstract numbers and arithmetic strategies. For example, children use their fingers to solve addition problems. The conceptual subitizing of recognizing the finger structures support children to do counting on. Children who can not subitize number structure might have a slow arithmetic development.

More precisely, Van Eerde (1996) has shown that using structures in fingers, egg boxes and math rack to help students move from counting all to an abbreviation of counting, such as counting by grouping. For example, the structures of an egg box can promote students to use groups of 5 or groups of 10. To tell that there are 8 eggs, students can reason by groups of 5 (i.e., “I know there are 5eggs in one row, and 3 eggs in the other row and there are 8 eggs altogether”) or by group of 10 (i.e., “10 eggs altogether and 2 eggs are missing”). And then gradually, students connect the structures to the formal mathematics such as 8 = 5 + 3 and 5 + 3 = 8 or 8 = 10 – 2 and 8 + 2 = 10.

Mulligan, Prescott and Mitchelmore (2004), have also strongly recommended using structures by assisting children to visualize and record patterns accurately. This approach might lead to a much broader improvement in children’s mathematical understanding. However, Mulligan, Prescott and Mitchelmore (2004) also noted that some young children do not develop understanding of structures while working with mathematical concepts. This then raised some questions such as why they do not use structures and how to promote them in using the structures.

2.5. Research questions

Inspired by the former research results, we are challenged to design classroom activities that promote students to structure and use the structures in their mathematical reasoning. Therefore, firstly, we propose to promote students’ awareness of structures in which they are learning to recognize structures and construct their own structures. Once students are accustomed to structures, then they can employ the structures for constructing more sophisticated counting strategies for example from counting all to counting by grouping. Finally, the structures and counting strategies will serve as the base for students to conceptualize the abbreviated strategies of solving additions up to 20. In this research we are looking for the answer of the following questions:

1. What kinds of structures of amounts up to 20 are suitable in the Indonesian context?
2. How does the role of structures evolve when students learn to abbreviate the strategies of addition up to 20?

Chapter 3

Research Methodology and Data Collection

3.1. Design Research

This research will be conducted under a design research methodology. Gravemeijer and Cobb (2006) explained that design research consists of cycles which have three phases: the preparation phase, the teaching experiment, and the retrospective analysis. In the preparation phase, a hypothetical learning trajectory is designed which consists of teaching-learning activities and conjectures of students' thinking processes based on the theorem that had been developed before. Next, the conjectures are tested in the teaching experiment. The goal of this experimental phase is to test and improve the conjectured learning trajectory and to develop an understanding of how it works. During this period, data such as classroom observations, teacher's interview and students' work will be collected and will be analyzed in the retrospective analysis phase. The result of the retrospective analysis will add to the local instruction theory and will give an evaluation of the initial hypothetical learning trajectory that contributes to the improvement of the next one.

According to Simon (1995) a hypothetical learning trajectory (HLT) is made up of three components, namely the learning goal, the learning activities, and the hypothetical learning process. The goal determines the design of the learning activities. In order to reach the goal, first, researchers need to set up the starting point of the learning, that is the current students' knowledge of the mathematical domain being taught. After that, activities are designed to help students achieve the goals. In designing the activities, the researchers need to make predictions of how students' students understanding will evolve throughout the activities.

Moreover, Simon (1995) added that the teacher-students interactions are also taken into account of the progressive process of designing a HLT. During teacher-students interactions, researchers observe how students learn and whether the learning meets the expectation in the HLT. These observations then constitute a refinement and improvement of the designed HLT. In line with Simon, Gravemeijer and Cobb (2006) emphasized that in each lesson, the researches should analyze the actual process of students' learning and the anticipated one. On the basis of this analysis, the researchers decide about the revision of the HLT.

Underpinned by the continual improvement of the HLT, the cyclic process is one of the main properties of design research. Cycles are always refined to form a new cycle in the emergence of a local instructional theory.

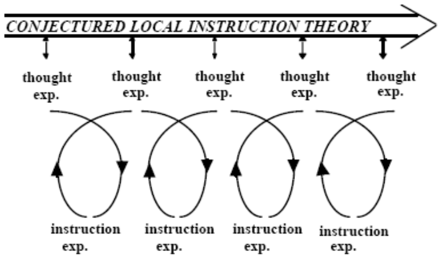


Figure 3.1: The cyclic process of design research (Gravemeijer & Cobb, 2006)

3.2. Data Collection

The research was conducted in an elementary school in Jakarta, Indonesia. The name of the school is SDN Percontohan Kompleks IKIP Rawamangun Jakarta Timur. The experimental class consisted of 37 students at the age of 7 to 8 years old. The experiment was divided into two parts; part 1 was conducted in the period of May to June 2008. During this period, the experimental class was at the end of grade 1. We concentrated on investigating student's current knowledge and testing some of the activities in our initial hypothetical learning trajectory. In order to do that, we carried out some observations and interviews with grade 1 students and the teacher. We analyzed the data from the observations and the interviews to improve the HLT.

The second part of this research was conducted at the beginning of the following school year, in July to August 2008. In this period the students were in grade 2. By this time, we had improved our first HLT and we tested the improved HLT (HLT II). We conducted 6 lessons in which we aimed at testing our conjectured learning trajectory and investigating students' thinking process.

The data were collected through interviews with the teachers and the students, classroom observations, and students' work. After that, we analyzed these data in the retrospective analysis. The outline of our data collection is represented in the following table

	Data collected	Goal
Part 1: preliminary experiment (Mei – June 08)	Classroom observation of grade 1 *Video recording*	• Finding socio norms and socio-mathematical norms • Finding students' current knowledge of addition up to 20
	Interview with grade 1 teacher *Audio recording*	• Finding students' current knowledge of addition up to 20
	Teaching experiment with 5 students • Flash card Games *Video recording* • Candy packing activity *Video recording, students' work* • The sum I know Worksheet *Video recording, students' work* • Assessment on Addition up to 20 *Video recording, student' work*	• Testing some activities • Investigating students' strategies of solving addition up to 20
Part 2: experimental (July – August 08)	Classroom observation 6 lessons *Video recording, students' work*	• Testing all activities in HLT II and investigating students' thinking process
	Final assessment *Students' work*	• Finding the effects on students cognitive of addition up to 20.

Table 3.1: Outline of data collection

3.3. Data Analysis

In the retrospective analysis, I extensively analyzed the data I have collected during the experimental phase. In the analysis, I compared the HLT and the students' actual learning based on the video recording. Firstly, I watched the whole video and looked for fragments in which students learned or did not learn what I conjectured them to learn. I also found some fragments in which the learning took place was not expected in the HLT. After that, I registered these selected fragments for a better organization of the analysis. I leaved out the parts of video that were not relevant to students' learning.

Secondly, I transcribed the conversations between the teacher and the students in the selected fragments. Then I started the analysis by looking at short conversations and students' gestures in order to make interpretation of students' thinking processes. I also discussed my interpretation of students' learning in some fragments with my fellow students.

I also used other sources of data such as teacher's interview and students' work to improve the validity of the research (data triangulation). After that, I asked for second opinions of the analysis from the expert, my supervisor. We discussed the analyses intensively and then I improved them.

The analysis of the lessons was done in two ways; analysis on daily bases and analysis of the whole series of lessons. In the daily bases, the analysis focused on how the activities support the intended students' learning. While in the whole lesson series analysis, we focused on the connections between the lessons to find out if the succession of the activities supports students' learning.

Finally, we drew conclusions based on the retrospective analysis. These conclusions focused on answering the research questions. We also gave recommendations for the improvement of the HLT, for mathematics educational practice in Indonesia and for further research.

3.4. Validity and Reliability

The validity concerns the quality of the data collection and the conclusion that is drawn based on the data. The data were collected throughout the learning activities that were designed to support students' learning of abbreviating strategies of addition up to 20. To guarantee the internal validity of this research, we used many sources of data, namely video recordings of classroom observations, teachers' interviews and students' work. Having these data, allow us to conserve the triangulation so that we can control the quality of the conclusions. Beside that, we also analyzed the succession of different lessons to test our conjectures of students' learning development. We conducted the research in a real classroom setting, therefore we could guarantee the ecological validity.

To improve the internal reliability of the research, we transcribed critical episodes of the video recording. We also involved some colleagues in the analysis of the critical learning episodes (peer examinations). We registered and recorded the data in such a way that it is clear where the conclusion came from. In this way, we took care the external reliability, the trackability of the research, and documented our analysis.

In this research, we carried out the first cycle of the design, therefore we made an extensive data analysis in which we elaborated the progressive design process of HLT I and HLT II. And after that, we compare the HLT II with the actual students' learning. Underpinned by this analysis, we could see what students have learned or not learned, and also make recommendation of how HLT II should be improved for further studies.

Chapter 4

Hypothetical Learning Trajectory

The aim of this research is to develop classroom activities that support students' learning of addition up to 20 by using structures. In order to achieve that goal, first we designed a hypothetical learning trajectory (HLT) which contains the learning goals, learning activities and conjectures of students' thinking process. In this chapter, we describe the starting point of the students, our learning goals, activities that allow us to reach the goals, and the conjectures of students' thinking in the HLT. We have two versions of HLT; the first one is called HLT I and it was designed before part 1 of the experiment and the second one is the revised version of HLT I, which is called HLT II.

In the part 1 or the experiment, we conducted classroom observation, interviews with the teacher and the students, and also tested some activities in this HLT to study how the activities work in a real classroom situation. From the observations and interview with the teacher, we gathered data about students' current knowledge and learning history. The video recording of the classroom activities enabled us to analyze students' thinking process.

As the result of the first part of the experiment, we got some insights about students' current learning and how the activities would work with the students. Having those insights allowed us to set the starting point of the learning and revise HLT I. This revision then resulted in the HLT II, which was tested in the second part of the experiment.

First, we start our discussion by describing the process of designing HLT I which includes the students' starting point, learning goals, learning activities, and conjectures of students' thinking process in it. After that we elaborate the analysis of the trial of HLT I and the re-designing process of HLT II. Finally, we describe HLT II and the progressive design process of HLT I and HLT II.

4.1. Hypothetical Learning Trajectory I

During the preparation phase of the design research we sketched out some potential contexts to be brought out in the classroom activities. The idea was first to construct students' awareness of structures as a base for using and manipulating structures in counting and addition up to 20. We thought of some contexts that might be powerful to support students' awareness of structures such as bus context, building context, bowling game context, and candy packing context. Each of those contexts has potential to enable students' learning of structure and addition up to 20. We chose the best context, which was suitable not only for one activity but also for the whole sequence of the learning trajectory.

In Indonesian school, teaching early arithmetic is started by counting and ordering number, and then students are led to addition and subtraction up to 10 before they do addition and subtraction up to 20. In most schools, addition and subtraction up to 10 are taught through counting. Once students have understood that addition is joining two sets of objects, which implies having more quantity and subtracting is taking away some objects from a set which implies having less quantity, students basically do addition and subtraction by counting. This approach results in students' behavior of using their fingers when adding and subtracting even when they work on larger numbers.

This research will use another approach, namely using structures. We will use structure of numbers as a base to shorten counting strategy; from counting all to counting by grouping. This implies a constitution of a new socio mathematical norm for teacher and students in a sense that counting can be done in a smart and more efficient way. Both teacher and students will learn to shift their paradigm from doing addition by counting or formal algorithm, to using more flexible and meaningful strategies such as friends of 10 and doubling.

Our departure point is students' current knowledge and ability. We assume that students are able to count and do addition and subtraction through counting.

Activity 1: Awareness of structure

In this activity, we presume that students are not familiar with structures. Therefore the goal of this activity is to develop students' awareness of structures that is when students come to an understanding of the need and importance of using structures to move from counting all to counting by grouping. We think, awareness of structures is an important base for students before they work with more structures in the next lessons. In this activity, students will be stimulated to recognize structures and construct their own structures.

We use a candy packing context to evoke the need of structuring objects in order to abbreviate counting processes. We think, candy packing is a good context because in Indonesia candies are sold in a plastic bag. For a buyer, sometimes it can be problematic to determine how many candies are in the bag. Thus we want to bring this problem in to the classroom and use it to generate students' awareness of structures.

Students will be asked to make a candy packing that enables people to immediately recognize the number of candies inside the packing. In this activity students will be opposed to a problem that will stimulate them to create a way of doing conceptual subitizing. The teacher will give them a plastic bag full of candies and we presume that by moving the candies, trying out several arrangements students will develop an awareness of using structures in counting. Students will work in groups, so that they will have an opportunity to discuss the arrangement of the candy packing. Students' design might vary but we conjecture students will use structures of groups of 5 or groups of 10 as they have learned 10 and 5 structures in grade 1.

This task will be followed by making a pictorial representation of the packing. Students will be asked to make a drawing of their packing. This drawing serves as a *model of* the situation which represents the problem students are working on (Gravemeijer, 2006), which later on would be used for the group presentation.

Activity 2: Group presentation

After each group has finished making their candy packing, then they will have to present their packing in front of the class. The goal of this presentation is to disclose the structures that students made and how they used the structures in their counting. Students will show the drawing and explain how they used the structures in their counting strategy. We hope this presentation will open a dialog among students in which they can compare and find the differences and similarities of the structures. After that, students can draw some conclusions of what advantages they can get from each structure.

After all groups have presented their work, the teacher starts a classroom discussion about which structure is the best. We expect, in this discussion students will come to a classroom agreement of choosing the best structure that allows them do a better counting. The teacher will play an important role during this presentation. She will orchestrate the discussion so that the attention is on the structures, for instance groups of 5 or groups of 10 and how those structures support counting process. The best structure will be the one that can immediately help people recognize the number of candies in the packing by conceptual subitizing. We conjecture students will choose groups of 5 or groups of 10 as the best structure. As the result of this activity, we expect students would understand the need of structuring objects for a better counting.

Activity 3: double structure

At this point, students should be able to recognize and construct structures, since they have learned about structures in the previous activities. Now, we want to focus on the double structure and we aim at exposing some double sums and constructing double structure. The double sums are introduced through a song called "*Satu ditambah satu*" or "One plus one". Indonesian children are very familiar with this song.

> *Satu ditambah satu sama dengan dua*
> One plus one equals two
> *Dua ditambah dua sama dengan empat*
> Two plus two equals four
> *Empat ditambah empat sama dengan delapan*
> Four plus four equals eight
> *Delapan ditambah delapan sama dengan enam belas*
> Eight plus eight equals six teen

The teacher will ask the students to sing this song together. While singing the song, students are asked to make a group of a particular number of person. This will connect the singing and the action which allows students to experience the double sums. We predict that there will be many physical activities such as students walk around and call out each other to make the group. This movement might ruin the structures in the group, therefore we ask the teacher to constantly encourage students to preserve the structures in their group. Teacher will ask a question such as "How can I know the number of students in each group easily?" We hope students will be stimulated to maintain the structure in their group.

After this physical activity, we give students a worksheet in which they will be asked to recognize double structure in a candy packing. We maintain the use of candy packing to preserve the consistency of the structures. Not only will students be asked to tell how many candies are in the packing, but also to give their reasoning about it. We hope students are able use groups of 5, groups of 10 or double arguments when determining the number of candies. As the result of this activity, students are expected to be able to tell immediately how many candies in a pack by conceptual subitizing and give the mathematical reasoning.

Activity 4: Flash card games

The aim of this activity was to find out what structures that students are familiar with. We presume students have known some structures such as dice structures and finger structures. In the game, we will use finger structure, dice structures and egg box structures. Students will be asked to tell the number represented in each card. They only have a few seconds to see the card, and thus they do not have the opportunity to count one by one. This game will force them to recognize and use the structures to do a fast counting. We assume, at first students might need more time, but as soon as they recall the structure, they will be able to play this game easily.

We expect students will not have any difficulties in recognizing finger structures since they might still use fingers while counting. Dice structure will not be hard for students either because in Jakarta, there are many children games played with dice. We anticipate students might find difficulties working with egg box structure, since it is not too popular for students, thus we will bring the real egg box in the classroom. By showing the real object to students we hope they will be able to recognize the structure in it.

Students might use addition reasoning when they see the finger and dice structure. The structures in an egg box give more choices for students to use either groups of 5 or groups of 10. For example *seven* can be seen as 5 + 2 or 10 – 3 (figure 4.1)

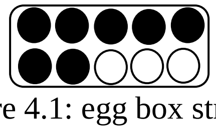


Figure 4.1: egg box structure

Activity 5: Finger structure

At this moment, we do not know specifically how students used their fingers for solving addition problems. Based on the information we got from the interview with the teacher, students are still using their fingers when working on mathematical problems, thus we assumed there are students who still counting all with their fingers. Therefore the aim of this activity is to promote students to use the finger structure smartly, so instead of counting all, we want students to be able to use conceptual subitizing.

Students will explore finger structures through a worksheet in which they will be asked to tell the number represented by the finger structures. We conjecture that there is a big range of students' ability of knowing and using finger structures. Some students might still do counting all, and some others might have been able to do advanced counting, for example by subitizing. We hope this worksheet will give students enough practice to be able to easily recognize and show a number by using their fingers.

Activity 6: ' The sum I know' worksheet

In this activity, we aim at generating students' understanding of number relations based on their current knowledge. We presume that students have not yet developed a number relation concept, that they still see additions individually and do not see the connection between numbers. As the result of this activity we hope students will be able to use some additions they already know by heart as a benchmark to solve other addition up to 20 problems. The worksheet activity is designed as a starter for a classroom discussion. Through the discussion, students will share their finding while working on the worksheet.

In "The Sum I Know" worksheet, students will be asked to circle all the sums they know and then write down the result of those additions. This activity will give students an opportunity to realize what they have known, and use the knowledge to move to other sums. For example, if a student knows that 3 + 5 is 8, we hope they will also know the neighbor of 3 + 5 for example 3 + 6 is 9 because 3 + 6 = 3 + 5 + 1 = 8 + 1 = 9.

We conjecture students will circle the double sums, since they had done some activities about double before. We will use these double sums to open a discussion in which students will learn a strategy of solving almost double sum, such as 8 + 7. If students have known that 7 + 7 = 14, they could easily solve 8 + 7 by doing 7 + 7 + 1 = 14 + 1 = 15.

Activity 7: Friends of 10

In this activity we aim at developing students' understanding of number pairs that make 10 or the friends of 10. We presume, at this moment, students are able to do conceptual subitizing. The context used now is to fill out a candy box. The candy box that is used has the same structure as the egg box structure. We do not use the egg box because we are afraid students are not familiar with it. Therefore we adapt the structure on an egg box and use it in the candy box.

The teacher will ask question such as "If there were only 7 candies in the box, how many more need to be put in?". This context can trigger students' understanding of number relation that makes 10 or the friends of 10. They should have been able to see the missing candies, there are 3 candies missing, thus 7 and 3 are the friends of 10.

Students will work on many other combinations of the friends of 10 through the candy box. We conjecture students might do this by using groups of 10, for example, since there are 10 candies in total, and 3 is missing, thus 7 candies are left. From this informal reasoning, the classroom discussion will take students to a more formal mathematics. In the discussion students will be guided to write their informal reasoning in a mathematical sentence. For example, students will be able to write $7 = 10 - 3$ or $7 + 3 = 10$. As the result of this activity, students are expected to understand the number pairs that make 10 or the friends of 10.

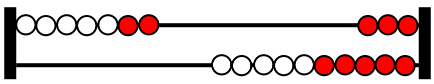
Activity 8: Addition up to 20

We assume that at this moment, students have understood the friends of 10. The goal of this activity is to stimulate students to use the friends of 10 in the decomposition to 10 strategy of solving addition up to 20 problems. In this activity, the friends of 10 will be introduced through a physical game. Students will form a group and they will play throwing disk game in which they will be able to do addition in a more active and physical way. They will be given two disks and they must throw the disks to a target. The targets will be numbered 1 to 9, and they must hit the biggest number in order to get bigger point. If the sum of the two disks is 10, they get a bonus of throwing one more time. This bonus will stimulate students to use the friends of 10.

We conjecture that they will try to hit 9 since it's the biggest number, and for the second throwing, they must hit 1 to get a 10 score for the bonus. However, if the first throwing did not hit 9, students can always choose the friends of 10 of that number. We hope this activity will allowed students to practice the friends of 10.

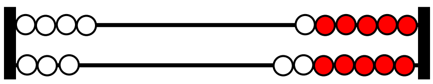
Activity 9: Addition up to 20 using math rack

By this moment, students should be able to find the friends of 10 as a step for doing decomposition to 10 strategy. In this activity, students will learn how to decompose the other addend to perform decomposition to 10 strategy. The goal of this activity is that students are able to perform decomposition to 10 strategy. In this activity we will introduce a new tool for doing addition up to 20; a math rack. Students will explore the structures in a math rack; we hope they will discover the structures of groups of 5 and groups of 10 in the math rack. First, students will be asked to show a number by using a math rack. Students might use different representations. For example *seven* can be represented as 5 + 2 or 10 – 3 or 4 + 3, etc.



7 = 5 + 2 or 7 = 10 – 3

Figure 4.2.a: Representation of seven



7 = 4 + 3

Figure 4.2.b: Representation of seven

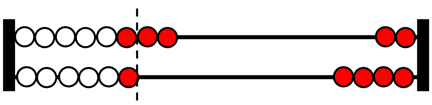
In figure 4.2.a, *seven* is 5 + 2 or it can also be 10 – 3, while in figure 4.2.b seven is 4 + 3. The teacher will use these differences to start a discussion about which structure is easier to recognize. We conjecture students will choose structures of 5 + 2 or 10 – 3 is easier than structure of 4 + 3.

After that, students will use the math rack as a tool to represent their strategy of solving addition up to 20. The teacher will give a problem, for example 8 + 6. Students will use the math rack to support their thinking. We conjecture some students might use the decomposition to 10 or double strategy.

8 + 2 + 4

10 + 4

Figure 4.3 : solving 8 + 6 by decomposition to10 strategy



6 + 6 + 2

Figure 4.4: solving 8 + 6 by double strategy

As the result of this activity, we hope students will be able to use double and decomposition to 10 strategy to solve addition up to 20 problems.

4.2. Analysis of part 1/preliminary experiment (May-June 2008)

In this section we elaborate the result of the first part of the experimental phase which was conducted during the period of May – June 2008. In this period, we visited the partner school and carried out some preliminary data collections (Table 3.1), such as classroom observations, interviews with the teacher and interviews with a small group of students. Throughout the classroom observations and interviews with the teacher we looked for what socio norms and socio mathematical norms have been developed in the classroom. From the interviews with the teacher we also gathered some information about the teacher's beliefs and students' current learning progress. More specifically, we looked for what students have learned especially about addition up to 20 which is specified into what strategies have been taught by the teacher and how she had taught those strategies. In addition, we worked with 5 students representing high, medium and low achievers in the class to get a deeper knowledge of students' strategies of addition up to 20. From this preliminary data collection, we get information to set up students' starting point and to revise HLT I.

4.2.1. Classroom observations

In the observations we found evidence of students' strategies of addition up to 20. The lesson was conducted when students were reviewing a lesson about number before a final test. The teacher gave some addition problems to the students and we found that students used different strategies. For example, students solved 8 + 7 in the following strategies:

<u>Solutions of 8 + 7</u>

$$7 + 7 + 1 = 15$$

Safira

$$\begin{array}{l} 8 = 0 + 8 \\ \underline{7 = 0 + 7} \\ \quad = 0 + 15 \\ \quad = 15 \end{array}$$

Laras

$$8 + 2 = 10 + 5 = 15$$

Faraz

$$\begin{array}{r} 8 \\ \underline{7}+ \\ 15 \end{array}$$

Haura

$$4 + 4 + 2 + 5 = 15$$

Janet

$$5 + 5 + 3 + 2 = 15$$

Gina

Most students did the addition by writing a formal notation (see Haura), but it did not represent their thinking. Even though they wrote a formal notation, they actually used different thinking process such as decomposition to 10, or even counting on with fingers.

Janet's answer is interesting, she decomposed the 8 in to 4 and 4, while the 7 into 2 and 5. She might know by heart that 4 and 4 make 8, in this case she used double. But she showed a different approach for 7. She couldn't use double because 7 is an odd number. She chose 5 and 2. When the teacher asked her why she used this strategy, she simply said "because it's easy". Unfortunately the teacher did not ask any further. Splitting the 8 into 4 and 4 did not really help her solving the problem. But the next step (i.e., splitting the 7 into 2 and 5) helped her shorten the calculation as 8 + 2 is 10, and 10 + 5 is 15.

Gina decomposed the 8 into 5 and 3, the 7 into 5 and 2. While explaining her answer she showed her calculation using her fingers.

> "we have 8, inside it there is 5, so we take 5 out. We have 7, inside it, there is 5 and we take that one out too. 5 and 5 is 10. We took 5 from 8, so now we only have 3, and 2 from the 7. 3 + 2 is 5. 10 + 5 = 15"

Gina's answer indicated that she understood hierarchical inclusion (i.e., understanding that there are numbers inside a number, six is inside seven) and used that concept to make groups of 5 and represented it with her fingers.

Safira and few other students used double strategy. These students knew by heart that 7 + 7 is 14 and 1 more is 15. While, Faraz and many other students used decomposition to 10 strategy. They decomposed the 7 into 2 and 5, therefore they could make the 8 into 10, and then it was followed by adding 5 to 10 which gave result 15.

Laras and Haura wrote formal notation when actually they still used counting on with their fingers. Laras's strategy was not suitable for addition up to 20 since that strategy was supposed to be used to solve addition of two digit numbers. Therefore, Laras was still doing counting on with her fingers. This behavior indicated that some students used the formal notation that is meaningless with numbers under 10, because it does not support students thinking.

The teacher asked each of these students to explain their strategy; unfortunately some students spoke very weakly that not all member of the class could hear. The teacher did not continue this session to a discussion of deciding which strategy is best for the class.

This observations in grade 1 led to some findings about the classroom socio norms. The classroom had a very open atmosphere where all students are allowed to use their own strategy. The students were also very enthusiastic and talkative that they did not afraid or shy in sharing their ideas and opinions. This good communicative culture will support our HLT since we had planned some discussion in the lessons.

We also found the socio-mathematical norms that have been established in the class room. Students used many strategies when solving mathematical problems, each strategy was always followed by an argument of how it worked. By giving the arguments, students explain their thinking process. However, this class hasn't built a norm that trains students to wisely choose the best strategy. Students tried out different strategies from informal to formal ones without having the awareness and ability to select one strategy that is most effective and most flexible. This condition did not encourage low level students to move from informal strategies to a more formal strategy since they are allowed to use the informal strategy.

Our observation showed that many students did not represent their actual strategy in their written work. These students most likely used counting on by fingers. For example, Dinda (7 years old), when solving a problem, she used a formal notation but actually still used her fingers.

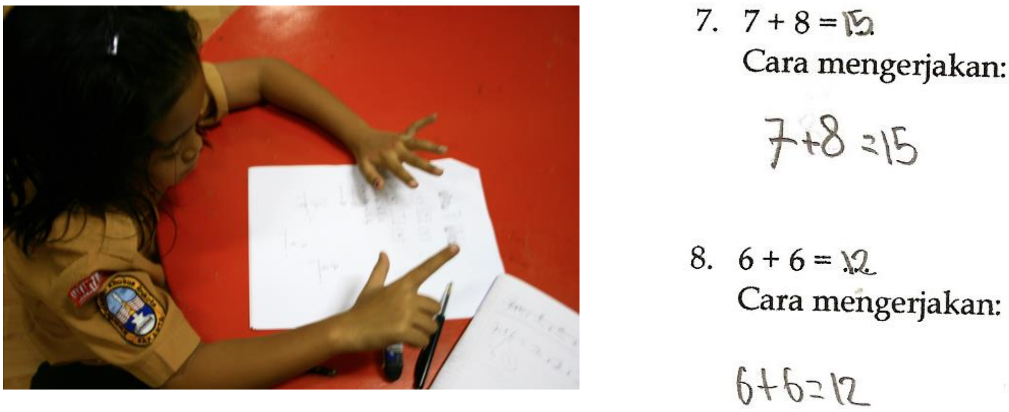


Figure 4. 5: Dinda's strategy: gap between formal written procedure and students' actual ability

This implies that in this class, a formal notation does not support the abbreviation of informal strategies. This was what Dinda experienced; she was still in the level of informal mathematics when the teacher has taught her formal mathematics therefore formal mathematics was meaningless for her.

4.2.2. Preliminary experiment

We tried out some of the activities in our initial HLT that are the key elements in the HLT. We work with 5 students and our investigation was focused on finding out what structures students know and how well they know those structures. Moreover, we wanted to find out what strategies used by the students when working on addition up to 20. The result of this trial will give us a feedback for the improvement of the HLT

Activity 1: Flash card game

In the first activity with 5 students, we tried out the flash card game in order to find out what structures students were familiar with. In the game, we used finger structures, dice structures and egg box structures. We predicted that students would not have any difficulties recognizing finger structures and dice structures, but the egg box structures might cause problems since students were not familiar with it. More precisely, we presumed students would not use counting all when determining the number represented by fingers and dice structure. They would use additions for these structures. For egg box structures, we conjectured that students used either addition or subtraction strategies, since the structures allowed them to do so.

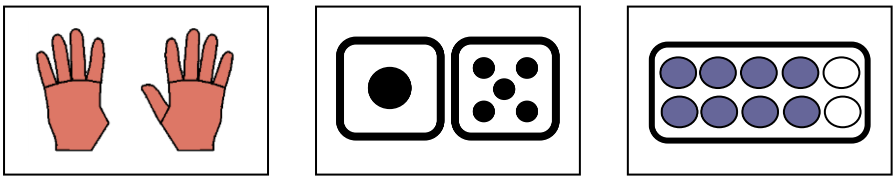
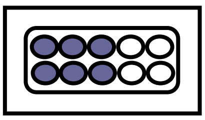


Figure 4.6: Flash card

The following is a segment from our video recording.

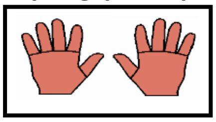
Researcher : (Telling the rules of the game. Showing the first card)



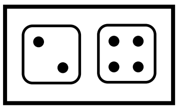
Dinda : 6
Researcher : How did you know that?
Dinda : I saw it and I know that 3 plus 3 is 6
Researcher : Ok, Now I'm going to ask the others, do you agree with Dinda?
Naga, Wira, Dini and Fathur response immediately by saying "yes"

Researcher : Does any of you have other ways of seeing the 6 black dots?
Naga : Because there are 10 dots, and I see 4 whites, so I know that there are 10 – 4 black dots, which is 6

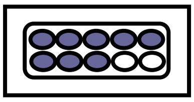
Researcher : Are you guys ready? (Showing another card)



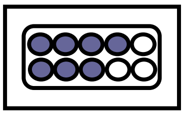
Naga : 2, (he quickly changed his answer) 10
Dinda : 10
Wira : 9
Fathur : 10
Researcher : Lets take a look at it one more time
All students said "ten" simultaneously
Researcher : What picture is this?
Students : Hands
Researcher : Show me with your hands, how many fingers are shown in this card?
Students raised their hands, showing 10 fingers
Researcher : Wira, why did you say two?
Wira : (Showing the right and the left hand) I thought there are two hands
Researcher : Ok, let's see the fingers. Now, I have another card. Ready?



Dinda : 6
Researcher : How could you know it so quickly?
Dinda : Because 4 + 2 is 6
Researcher : Ok, good. Where can you find this picture?
Students : Dice
Researcher : Do you play with dice often?
Students : Yes
Researcher : What kind of game?
Students : *Ular tangga* (snakes and stairs), monopoly
Researcher : Do you want to see other card?



Naga : 13
Fathur : 7
Wira : 7
Naga : 8
Dinda : 9
Researcher : Ok, let's take a look at it, how many are there?
Students : 8
Researcher : How come?
Wira : Because there are 5 and 3, so together are 8
Dinda : Because there are 2 whites, an all together are 10, so the black one is 10 – 2 equals 8
Researcher : (Showing another card)



Wira : 8
Researcher : How many are there?

Naga	:	(Counting one by one) 7
Dinda	:	I was looking at the whites, there are 3 whites, so 10 – 3 is 7
Dini	:	I was also looking at the whites. I know that in each row there are 5 dots, in the first row, I see 1 white, and it makes 4 black. In the second row I see two whites, thus 3 black. So 4 + 3 = 7

Based on our observations, we can make the following conclusions. Throughout this game, students have shown a good knowledge of structures. They were already familiar with the dice structures as they play a lot of games using dices such as monopoly and *ular tangga*. Students did not have any difficulties in determining the number of dots in two dices, however, sometimes they seemed not really sure of their answer, which probably caused by their over excitement of the game.

The finger structures were not a problem either; they could easily determine the number of fingers shown without counting. However, they seemed to have difficulties in determining the egg box structures because it was not familiar for them. In our next HLT we need to bring the real egg box into the classroom so that students could explore its structure. Based on students' explanation, most of the time students used additions or subtractions in answering the egg box structure. For example, when having the 8 card, student answered 5 + 3 or 10 – 2. This indicates they have a good understanding of additions and subtractions up to 10. Another evidence for this finding is when students are given the 7 card.

Students saw the whites, so 10 – 3 is 7 but they did not use double strategy. 3 + 3 + 1 = 7 was not mentioned by the students. This indicated that students were not used to work with doubles. The teacher interview signified this finding. She admitted that she only taught students the decomposition to 10 strategy.

Activity 2: Candy packing

We also tried the candy packing activity to test if the context is suitable for the students. We found that the candy problem has stimulated students to construct structures. Students were given two kinds of candy in one plastic bag, and were asked to tell how many candies were in the bag. We deliberately gave students two kinds of candies to see if it influenced students' grouping strategy. We conjecture students might use the kind of candies as a way of grouping.

At first, they took wild guesses and then to prove their guess, they counted the candies. At first students count the candies one by one, they found that there are 32 candies. When challenged to find a faster way of figuring out how many candies, each student come up with their own structures.

Naga made groups of 10. He put 10 candies in a row, there are 3 rows and 2 more candies, so all together there are 32 candies.

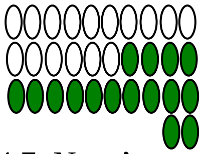


Figure 4.7: Naga’s arrangement

Wira said that he had a different way of arrangement. He divided the candies into two groups based on the type of the candies; Mentos and Kis. This proved our conjecture that students use the candy type as a way of grouping. Therefore by having only two types of candies students were promoted to use double structure. He arranged 10 Mentoses in a row, and 6 more mentoses in the second row. Then he puts 10 kises in the third row, and 6 more in the last row. He had 16 mentoses and 16 kises, so all together are 32.

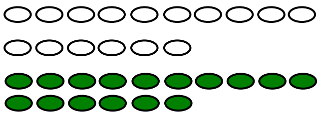


Figure 4.8: Wira’s arrangement

Naga and Dini Agreed with Wira by also saying 16 + 16 = 32. To assure he was correct, Naga counted the candies one by one, when he finally got 32, he was confident of his answer. But a problem arose; since Naga were still counting one by one, Wira was challenged to convince his peers that his arrangement is still good enough. However, before Wira explained his answer, Dinda interrupted by showing her arrangement.

Dinda made groups of 3 candies. However it was not clear why she chose groups of 3, when asked to explain her reasoning, she did not answer. After she finished her arrangement, she explained that she just added 3 and 3 and 3 until she found 30, and then 2 more gave her 32. Fathur supported Dinda’s argument by saying that there were 10 groups of 3 which make 30, and there were also 2 more candies, so there were 32 all together.

Wira tried another arrangement. He grouped each 4 candies into 1 group. He put each 4 Mentoses and 4 Kises together then he explained that 4 Mentoses and 4 Kises make 8 candies. Next, he added 8 and 8 and got 16, and 16 and 16 was 32. He explained verbally while adding 16 and 16, that 6 + 6 is 12, he split 12 into 10 and 2. Then he added all the tens together and got 30, and he still had 2 ones, which finally resulted 32. The way he did the addition showed that he used a formal algorithm of splitting tens and ones and adding them separately. This indicates that he is used to work with formal algorithm in solving addition problems.

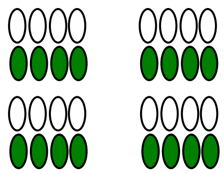


Figure 4.9: Wira’s arrangement

Dini made groups of 5 candies. She made dice structures, this indicated the previous activity; the flash card games has had a great influence on her thinking. She showed that 5 and 5 is 10, and 5 more is 15, and 5 more is 20. There are still 10 more so in total are 30 candies. She had 2 single candies, which resulted 32 candies.

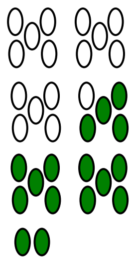


Figure 4.10: Dini’s arrangement

After all students tried out their candy arrangement, they were asked to determine which strategies is the easiest in finding out the number of candies. All students responded spontaneously that the groups of 5 allowed them to determine the number of candies easily. This indicated that through some explorations, students made comparisons of the groupings. They experienced the counting process and they found that groups of 5 is easies for them.

Our main intention was to get students use groups of 5 or groups of 10, however in the actual activity, student used other groups such as groups of 2, groups of 3 and groups of 4. The number of candies given, which is 32, has made students use those groups, thus in our next HLT we would only use 20 candies with two different flavors. The distinction of two flavors of candies can be used to promote students in using doubles. In this case, since we only have 20 candies, double also means groups of 10.

Activity 3: "The Sum I Know" worksheet

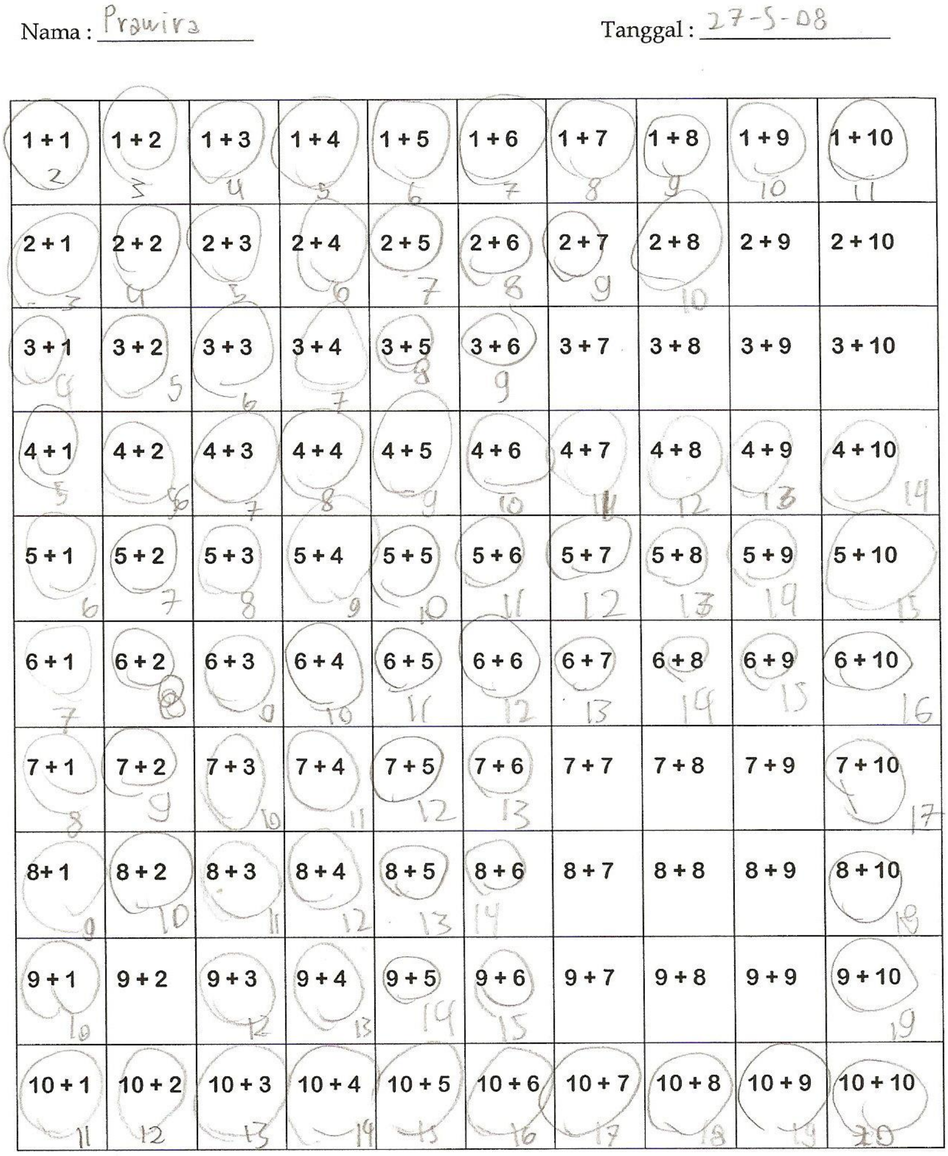
In this session, we tried out the "The Sum I Know" worksheet to find out how far students have known addition up to 20. Students were asked to circle all the sums they know by heart in only 15 minutes. We conjectured they would circle the first row, the first column, and some friends of 10 additions.

The observation showed, first, most students circled all the sums in the first row, the second row, the first column, the last column and sums that make 10 (Figure 4.11). However, Naga seemed to know all additions by heart since first, he circled the first row, and then he moved on to the second, third, etc. By the time he finished circling all the sums, he wrote the result of the sums. He did it all without having any difficulties. This indicated that Naga has mastered addition up to 20 very well.

Before time was up, Wira and Fathur finished circling all sums they know. They circled the first row and the last row, the first column, the last column and some additions in the upper to middle row especially the sums that make 10. Then they were asked to write the result of all the sums they had circled. After doing this, they discovered something; they found out that the results of the sums in one row were in order. Wira and Fathur discovered that every time they moved one step to the right, the sums always increased by one. They shouted out "I know all the sums" and then they immediately circle the other sums in the table.

Dinda circled almost all of the sums but when asked to write the result of the sums, she only wrote few of them. She saw Naga circled all the sums and she followed him without knowing the results. Clearly, Dinda felt insecure that she did not want to show her incompetence and she could do as good as the others. Dinda needs support to build her self esteem. Dini worked very quietly and slowly but produced a very good and precise result. She circled and wrote all the result of the sums.

We used Wira and Fathur' discovery to raise a short discussion. Students were asked to solve 7 + 7, they immediately answered 14. Then it was followed by 7 + 8, they answered 15 because 7 + 8 was next to 7 + 7. It implies 7 + 8 = 7 + 7 + 1 = 14 + 1 = 15. Wira and Fathur discovered the number relations within the worksheet. For our next HLT, this discovery was a critical learning moment that can be brought into a classroom discussion. This discussion will give an opportunity for students to constitute number relations.



Nama : Prawira Tanggal : 27-5-08

1 + 1 2	1 + 2 3	1 + 3 4	1 + 4 5	1 + 5 6	1 + 6 7	1 + 7 8	1 + 8 9	1 + 9 10	1 + 10 11
2 + 1 3	2 + 2 4	2 + 3 5	2 + 4 6	2 + 5 7	2 + 6 8	2 + 7 9	2 + 8 10	2 + 9	2 + 10
3 + 1 4	3 + 2 5	3 + 3 6	3 + 4 7	3 + 5 8	3 + 6 9	3 + 7	3 + 8	3 + 9	3 + 10
4 + 1 5	4 + 2 6	4 + 3 7	4 + 4 8	4 + 5 9	4 + 6 10	4 + 7 11	4 + 8 12	4 + 9 13	4 + 10 14
5 + 1 6	5 + 2 7	5 + 3 8	5 + 4 9	5 + 5 10	5 + 6 11	5 + 7 12	5 + 8 13	5 + 9 14	5 + 10 15
6 + 1 7	6 + 2 8	6 + 3 9	6 + 4 10	6 + 5 11	6 + 6 12	6 + 7 13	6 + 8 14	6 + 9 15	6 + 10 16
7 + 1 8	7 + 2 9	7 + 3 10	7 + 4 11	7 + 5 12	7 + 6 13	7 + 7	7 + 8	7 + 9	7 + 10 17
8+ 1 9	8 + 2 10	8 + 3 11	8 + 4 12	8 + 5 13	8 + 6 14	8 + 7	8 + 8	8 + 9	8 + 10 18
9 + 1 10	9 + 2	9 + 3 12	9 + 4 13	9 + 5 14	9 + 6 15	9 + 7	9 + 8	9 + 9	9 + 10 19
10 + 1 11	10 + 2 12	10 + 3 13	10 + 4 14	10 + 5 15	10 + 6 16	10 + 7 17	10 + 8 18	10 + 9 19	10 + 10 20

Figure 4.11: The Sum I Know worksheet

After trying out this activity, we drew some conclusions. We should reconsider the time for this activity, 15 minutes was too much. To find out students initial knowledge of additions, they should be given no more than 5 minutes. Based on the observations, students worked from the first row and then go to the second row, etc, thus, they only found the patterns of getting one more each time they move down or move to the right. Students and teacher can explore more from this activity.

The teacher can raise new problems, such as what if they go from right to left? Jump 5 steps?

Activity 4: Addition up to 20

To find what kind of strategies students normally use in solving addition problem, we asked them to do a worksheet in which students worked on a written addition word problem. We wanted to see weather students were able to translate real life situation given in word problems in to a mathematical expression. Students were allowed to solve the problem by using their own strategy, and we presumed that students would use different strategies such as double and decomposition to 10 or even counting on with fingers.

The first question was:

> "Susan loves to read, she's reading a story book about Malin Kundang. Yesterday she read 9 pages, and today she reads 7 more pages. At what page is she now?"

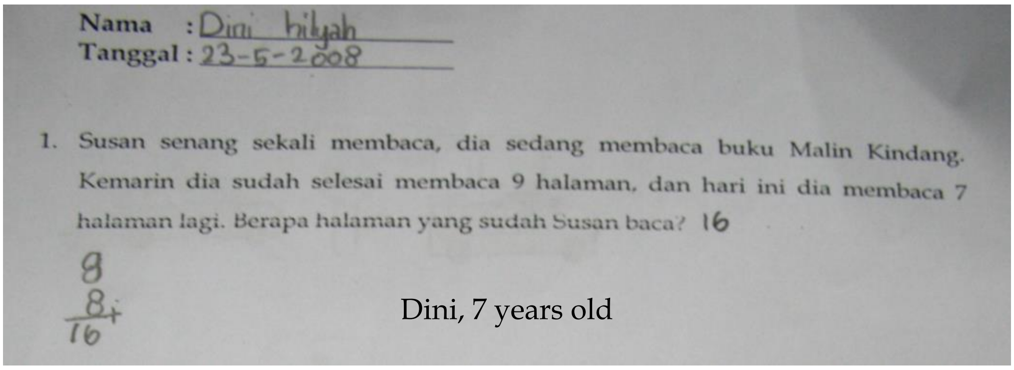
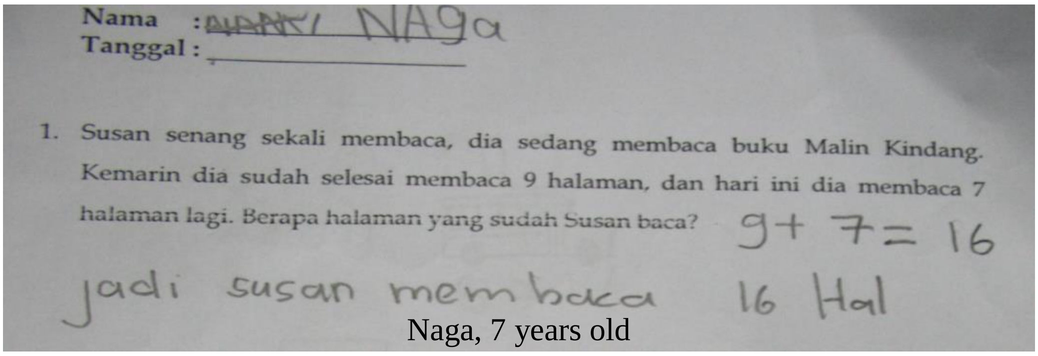


Figure 4.12 : Dini's addition strategy

Dini used decomposition to 10 strategy, she explained her strategy orally, she kept the 9 hold, and broke the 7 into 1 and 6. The 9 and the 1 made 10. 10 and 6 made 16. After that, she changed her strategy, now she used double, this time she kept the 7 hold, and she decomposed the 9 into 1 and 8. The 7 and the 1 became 8. 8 and 8 made 16. She explained this orally, and only wrote "8 + 8 = 16" in her worksheet. This is an interesting informal strategy. Dini performed a good understanding of using either double or decomposition to 10 strategy. She also showed an ability of splitting a number into two smaller numbers and make connections to the other number to get to a benchmark addition such as 10, and use that benchmark to help her solve the problem.

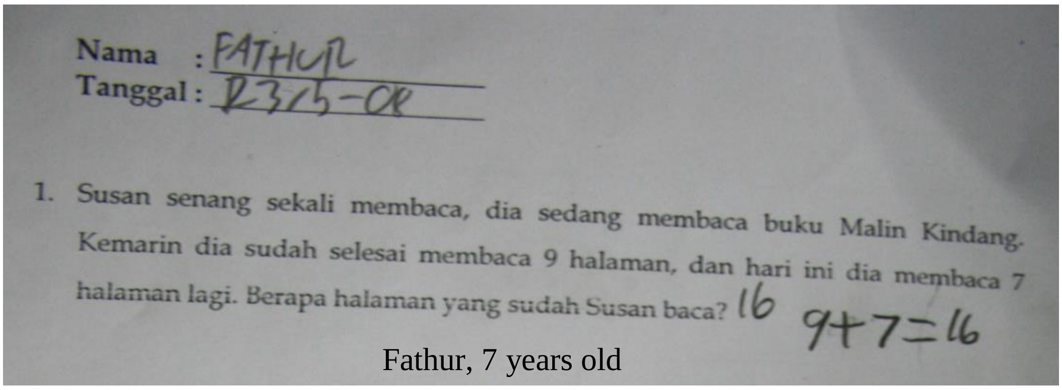

Nama : NAga
Tanggal :

1. Susan senang sekali membaca, dia sedang membaca buku Malin Kindang. Kemarin dia sudah selesai membaca 9 halaman, dan hari ini dia membaca 7 halaman lagi. Berapa halaman yang sudah Susan baca? 9+ 7= 16
jadi susan membaca 16 Hal

Naga, 7 years old

Figure 4.13: Naga’s addition strategy

Naga used formal notation, we assumed he knows this addition by heart.

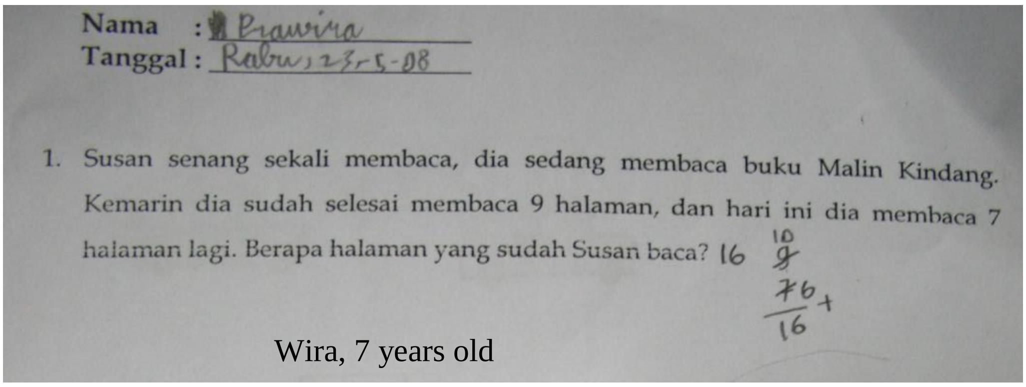

Nama : FATHUR
Tanggal : 23/5-08

1. Susan senang sekali membaca, dia sedang membaca buku Malin Kindang. Kemarin dia sudah selesai membaca 9 halaman, dan hari ini dia membaca 7 halaman lagi. Berapa halaman yang sudah Susan baca? 16 9+7=16

Fathur, 7 years old

Figure 4.14: Fathur’s addition strategy

Fathur used decomposition to 10, however the strategy was not written explicitly in his worksheet. He explained orally that first he added 1 to the 9 to get 10, and he had 6 left from the 7, 10 + 6 = 16.

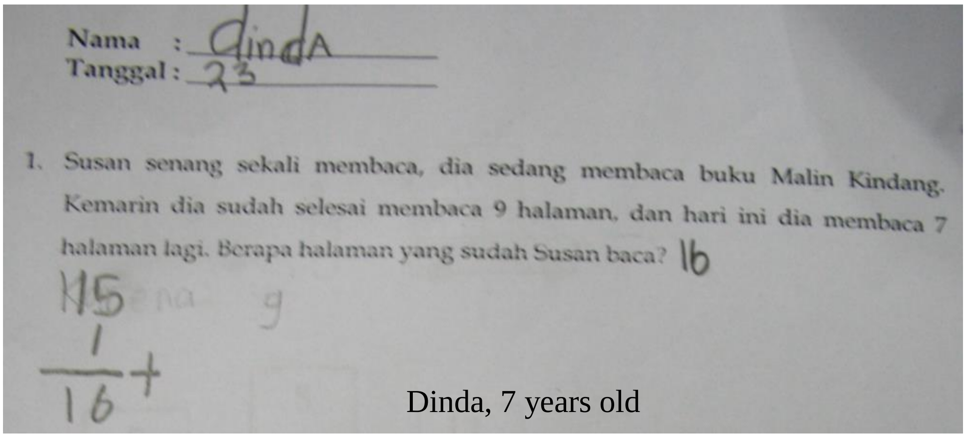

Nama : Prawira
Tanggal : Rabu, 23-5-08

1. Susan senang sekali membaca, dia sedang membaca buku Malin Kindang. Kemarin dia sudah selesai membaca 9 halaman, dan hari ini dia membaca 7 halaman lagi. Berapa halaman yang sudah Susan baca? 16 ~~9~~ 10
~~7~~ 6 +
16

Wira, 7 years old

Figure 4.15: Wira’s addition strategy

In his written work, Wira used decomposition to 10 strategy. He did the compensation by taking away 1 from 7 and adding it to the 9. He got 10 + 6 = 16.


Nama : dindA
Tanggal : 23

1. Susan senang sekali membaca, dia sedang membaca buku Malin Kindang. Kemarin dia sudah selesai membaca 9 halaman, dan hari ini dia membaca 7 halaman lagi. Berapa halaman yang sudah Susan baca? 16

15
1
16 +

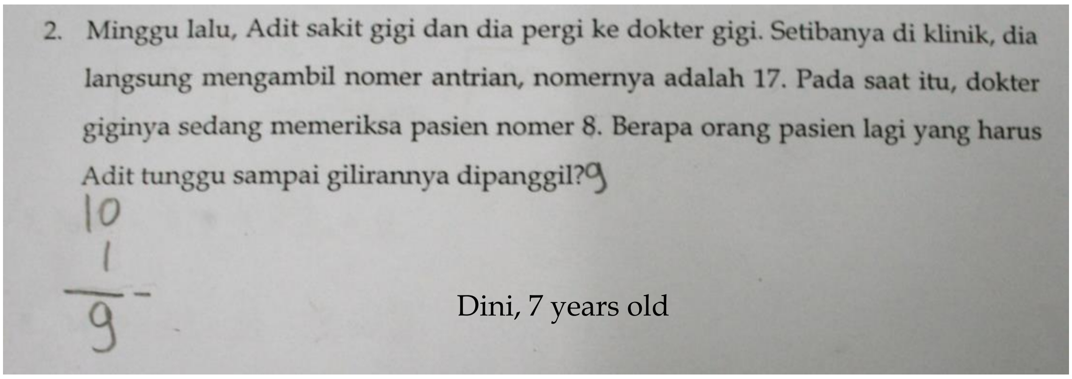
Dinda, 7 years old

Figure 4.16: Dinda’s addition strategy

Dinda was easily distracted, and was reluctant to read the question, therefore she did not know what to do with it. She looked at Dini’s work and she found out that the answer is 16, she just had to use different way to get to 16, and she wrote 15 + 1. Dinda was clearly did not understand the question. After she copied Dini’s answer, she just needed to find different way of getting 16, the easiest answer possible is 15 + 1.

The second question was still about addition but given in a different context.

> “Last week Adit went to a dentist because he’s been having a toothache. When he arrived at the dentist, he took a queue number, and his number was 17. At that time the doctor was having patient number 8 in his room. How many more patients did Adit have to wait before his turn? “

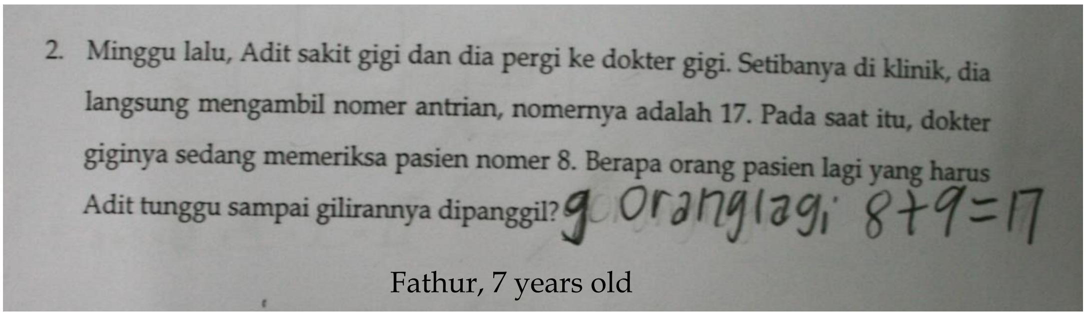

2. Minggu lalu, Adit sakit gigi dan dia pergi ke dokter gigi. Setibanya di klinik, dia langsung mengambil nomer antrian, nomernya adalah 17. Pada saat itu, dokter giginya sedang memeriksa pasien nomer 8. Berapa orang pasien lagi yang harus Adit tunggu sampai gilirannya dipanggil? 9

10
1
9 −

Dini, 7 years old

Figure 4.17: Dini’s strategy

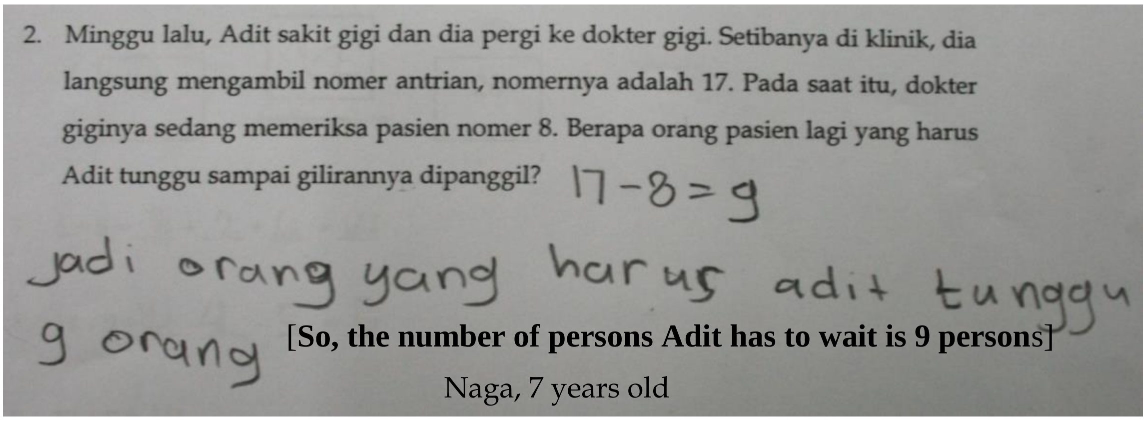
Dini said "what should be added to 8 to make 17?" She broke the 17 into 10 and 7, now she had 7 and 8, she canceled the 7 and she got a remainder 1. Then, she subtracted the 1 from the 10 which gave her result 9.


2. Minggu lalu, Adit sakit gigi dan dia pergi ke dokter gigi. Setibanya di klinik, dia langsung mengambil nomer antrian, nomernya adalah 17. Pada saat itu, dokter giginya sedang memeriksa pasien nomer 8. Berapa orang pasien lagi yang harus Adit tunggu sampai gilirannya dipanggil? 9 orang lagi 8+9=17

Fathur, 7 years old

Figure 4.18: Fathur's strategy

Fathur thought of the problem as an addition.


2. Minggu lalu, Adit sakit gigi dan dia pergi ke dokter gigi. Setibanya di klinik, dia langsung mengambil nomer antrian, nomernya adalah 17. Pada saat itu, dokter giginya sedang memeriksa pasien nomer 8. Berapa orang pasien lagi yang harus Adit tunggu sampai gilirannya dipanggil? 17-8=9

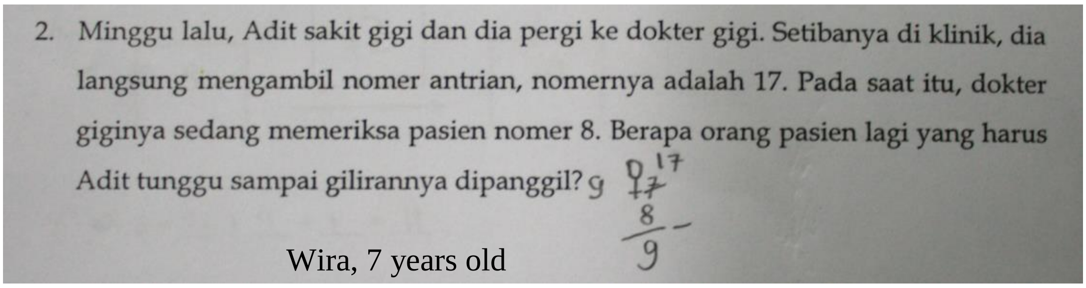
jadi orang yang harus adit tunggu 9 orang

[So, the number of persons Adit has to wait is 9 persons]

Naga, 7 years old

Figure 4.19: Naga's strategy

Naga straightly used subtraction to solve this problem. His written work indicated that he has no difficulties understanding the question. When asked to explain his thinking process, he just said that he knew it.


2. Minggu lalu, Adit sakit gigi dan dia pergi ke dokter gigi. Setibanya di klinik, dia langsung mengambil nomer antrian, nomernya adalah 17. Pada saat itu, dokter giginya sedang memeriksa pasien nomer 8. Berapa orang pasien lagi yang harus Adit tunggu sampai gilirannya dipanggil? 9

0 17
~~17~~
8 -
9

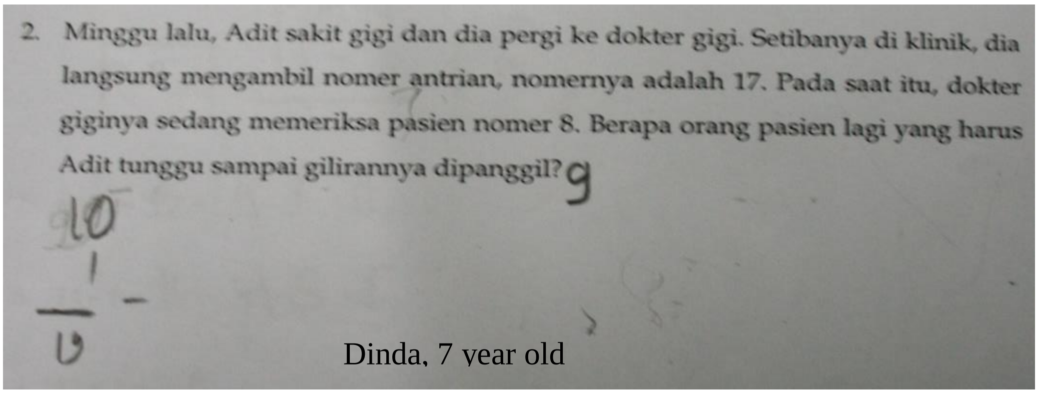
Wira, 7 years old

Figure 4.20: Wira's strategy

Wira used a formal notation; he solved subtraction up to 20 by using an algorithm. When asked to explain his thinking he said:

> "first, 7 – 8, we can not do that, so we borrow 1 from the tens. So now we have 17, 17 – 8 is 9"

His explanation indicated that he has understood place value and formal algorithm. However, he did not know when to use the formal strategy and he was not used to work with informal strategy.


2. Minggu lalu, Adit sakit gigi dan dia pergi ke dokter gigi. Setibanya di klinik, dia langsung mengambil nomer antrian, nomernya adalah 17. Pada saat itu, dokter giginya sedang memeriksa pasien nomer 8. Berapa orang pasien lagi yang harus Adit tunggu sampai gilirannya dipanggil? 9

Dinda, 7 year old

Figure 4.21: Dinda's strategy

Dinda was frustrated when working with this problem. Her concentration was distracted most of the time. After a while, she managed to calm down and work on the problem but her unwillingness to read the question has caused her difficulties to translate the problem into a mathematical argument.

Based on students written works, we can draw some conclusions. Students used different strategies in solving the problem. However, students in this class were not used to write their strategy. The answer written in the worksheet does not necessary show the real thinking process. Dini performed a very good informal addition strategy but she did not write it clearly. Fathur gave a clear verbal explanation but not written. This indication might have been caused by the absence of socio-mathematical norms. Students were not encouraged to write their real thinking process but they wrote a standard notation because that was what they thought they were supposed to do.

4.2.3. Conclusion of the Preliminary Experiment

The observations showed that the candy packing context is good to evoke students' awareness of structures, however, we need to consider the number of candies to anticipate students' grouping. In this experiment, 32 candies has promoted them to use many groupings, therefore in the next HLT we will only give students 20 candies to minimize the possibilities of groups used by the students.

"The sum I know" activity has stimulated students to discover number relations. The number relation concept will help students solve almost double sums. Therefore we will maintain this activity for the next HLT with a small adjustment. Instead of giving students 15 minutes to fill out the worksheet, we will only give them 5 minutes, so that we can minimize the possibility for them to count.

During this period, we found a large gap between high achieving students and low achievers. The high achievers have shown a mathematical growth as they were able to use more abbreviated strategies. However, the lower achievers were still struggling moving from informal to more formal mathematics. The socio-mathematical norms that we observed in this classroom might have caused this problem. Students were allowed to use their own strategy and were not stimulated to use more effective and flexible strategies. The low achievers felt comfortable using a counting on strategy, and therefore they were not promoted to use an abbreviated strategy.

4.3. Hypothetical Learning Trajectory II

During the first part of the experimental phase, we have conducted a preliminary experiment of some activities in the first HLT. After having the results of the experiment, we made some adjustments and improvements of the HLT. The revised version of the HLT is then called the HLT II. In this section we will describe the activities in HLT II. Some of the activities in HLT I are not changed, some are canceled, and some are improved. For activities that are not changed, we do not repeat the description, but refer to HLT I. Nevertheless, we will give a detail description for new activities.

Activity 1: Candy packing

From the preliminary experiment, we found that candy packing context is very useful in evoking the need of structuring to do a better counting. Children and candies are very closely related and we though the candies will attract students' attention. Moreover, the candies are also very hands-on that students can touch, move, arrange and rearrange. We made some adjustments based on the result of the preliminary experiment. We will deliberately use only 20 candies with 2 different flavors to minimize the possibility of students using other structures than group of 10 or group of 5.

Activity 2: Group Presentation (refer to activity 2 HLT I)

Activity 3: Flash card games (refer to activity 4 HLT I)

Activity 4: Double structure through singing (refer to activity 3 HLT I)

Activity 5: Double structure using candy packing

At this moment, students should have been able to pick up some double sums from the double song. In this activity, our objective is to relate those double sums with structured visualization. To maintain the consistency, we will use the group of 10 candy packing.

Students will work on a worksheet in which they will find an empty candy packing. They will be asked to fill out some candies in the packing by coloring the worksheet. Students strategies of structuring the candies can be seen by the coloring pattern they made. We conjecture the song will influence students in coloring the worksheet that they would use double structure.

After coloring activity, students will have another worksheet. In this worksheet, they have to determine the number of candies given. Students will also be asked to explain their counting strategy. Here, we expect students used double strategy in their counting.

Activity 6: Number relation through "The sum I know" worksheet

From the previous activities we hope students have developed an understanding of double structure. In this activity, our main goal is to construct an addition framework that use students' current knowledge of addition as a point of reference to explore more additions close to it. By this moment, students should have known some double sums; therefore we will use them as the points of reference. For example, if students know that 8 + 8 = 16, they can use it to determine 8 + 9, that is 8 + 8 + 1 = 16 + 1 = 17.

Students will have a worksheet, in which they will be asked to circle all the sums they know by heart. We expect students will circle some double sums which were sung in the song. These double sums will be the point of reference for students to do "*almost double"* additions such as 6 + 7, 7 + 8, etc. Students will be given 5 minutes to circle as many as additions they know. After that, the teacher will ask them to stop, and write the result of the addition that they had circled. By writing the result, we hope student will discover a pattern of addition framework, that each time they move one step to the right; the addition gets bigger by one.

We anticipate that not all students could come up with the framework of addition, thus we will hold a class room discussion. We provide a large "The sum I know" worksheet and put it on the white board in front of the class. Some students will be asked to circle the worksheet, and the teacher will use the circled additions to start a discussion. Teacher would pose a question such as "You have circled 6 + 6, and who can circle 6 + 7?, how did you know that?". We conjecture that some students will use the addition framework, 6 + 7 = 6 + 6 + 1 = 13. In this discussion, students will share their strategy and we hope more students would come to an understanding of using framework of addition to solve "*almost double*" problems.

Activity 7: Friends of 10 through egg box structures

In this activity we focus on the friends of 10 strategy. We will use egg box structures to develop students' understanding of the friends of ten. First students will be asked to tell how many eggs are in the box and give a reasoning of their counting strategy. We conjecture students will use addition or subtraction to shorten the counting. For example, "there are 8 eggs, because 5 eggs are in the first row, and 3 eggs in the second row" or "there are 8 eggs because 2 eggs are missing". In the first statement, students used groups of 5 while in the second, they used groups of 10.

Next, students will work on a worksheet in which they have to fill out the number of black dots (representing eggs) and the number of white dots (representing missing eggs). Each pair of blacks and whites will make 10 when added together. The teacher will drive a small discussion so that students will be able to conceive this idea. From this worksheet, we hope students will come to an understanding that the number of black and white dots together makes 10.

Activity 8: Friends of ten through finding friend games

By this moment, we conjecture that students should have been able to find the friends of 10, for example 6 and 4, 7 and 3, etc. Our goal in this activity is to strengthen students understanding of the friends of 10. We design a game where students have a number and they must find a friend so that together they would make 10. We conjecture that students would not have any difficulties doing this game. Moreover, we predict some students will find their friends by doing counting on or some students might have known the friends of 10 by heart.

Students will not only find the friends of 10, but also other numbers less than 20. For example, when the teacher writes 17 on the white board, each student have to find one or two friends to make 17. During this game, students will be constantly doing additions; we hope this could be a good practice for them.

Activity 9: Addition up to 20 using math rack (refer to activity 9 HLT I)

4.4. Progressive Design of HLT I and HLT II: a Summary

To sum up, we describe the progressive design of HLT I and HLT II in the following table:

HLT I		HLT II
Activity 1: Candy packing		Activity 1: Candy packing Adjusted by only use 20 candies
Activity 2: Group Presentation		Activity 2: Group Presentation
Activity 3: Double song		Activity 3: Flash card games
Activity 4: Flash card games		Activity 4: Double song
Activity 5: Finger structures	Canceled	Activity 5: double structure in a candy box Improved with a worksheet
Activity 6: The Sum I know worksheet		Activity 6: The Sum I know worksheet Adjusted by only giving 5 minutes for students to work on the worksheet
Activity 7: Friends of 10 in a candy box	Canceled	Activity 7: Friends of 10 in an egg box
Activity 8: Throwing disc game	Canceled	Activity 8 : Friends of 10 through finding friend games
Activity 9: Addition up to 20 by using a math rack		Activity 9: Addition up to 20 by using a math rack

Table 4.1: the progressive design of HLT I and HLT II

Chapter 5

Retrospective Analysis

In this chapter, we compared our HLT II and students' actual learning process during the experimental phase. We investigated if and how the HLT supported students' learning. First, we looked at the video recordings, and selected some critical moments in which students learned something or students did not learn as was expected in the HLT. Then we transcribed these critical moments we have observed in the classroom. These transcriptions were the empirical bases for our interpretations of student's learning processes. We also analyzed students' written work as another source to investigate students' learning. Moreover, we discussed what made successful activities and what students have learned from those activities. In the case of unsuccessful activities, we investigated what caused such failure, and what needed to be done in the next HLT to improve students' learning processes.

We should point out that during the teaching-learning experiment, we followed, observed and studied each lesson to find out whether the actual students' learning process met the expectation in the HLT. Therefore, we made changes and added some activities on daily basis to adjust and improve students' learning.

We analyzed the data in two ways. First we analyzed the day by day lessons. In this analysis we focused on what happened with the students, the teacher, the activities, the materials used, and how each of these contributed to the lesson. In the second analysis we looked with a broader view and searched for connections between the lessons and tried to find out how earlier lessons supports the following ones.

The result of the retrospective analysis will be used to improve our HLT II and design the HLT III, and to answer our research questions.

5.1. Lesson 1: Awareness of structure (6 August 2008)

In our HLT II, our first goal was to generate students' awareness of structures, that is when students can recognize structures and use them in reasoning for an effective counting strategy. An effective counting strategy is a way of counting that does not rely on counting all, but make use of the structures which results in an abbreviated and more accurate counting. Our first lesson was designed to evoke the need of using structures to do an effective counting, for example when students can move from counting one by one to counting by grouping.

We chose the candy packing context because in Indonesia candies are sold in a plastic bag which often causes problems for a buyer who wants to know how many candies are in the bag. Through this problem, we hoped to stimulate students to think of an easy way of determining the quantity of the candies in the packing. Students played as the candy seller who needs a new packing so that his/her customer can determine the number of the candies easily. We hope, when playing as a seller, students would be fully engaged and stimulated to discover a creative way of arranging the candies.

The candies were given in a plastic bag which consisted of 20 candies in 2 different flavors. Students would be asked to determine the number of the candies in the plastic bag. We deliberately chose 20 candies in 2 different flavors which have different colors, to anticipate the using of all kind of groups by the students. Our intention was that students would use groups of 10 and groups of 5, and we thought 20 candies would allow students to do so and it also minimized the possibility of using other groups. We predicted, students would use counting all since they only have 20 candies. The problem to create a candy packing that help costumers to know the number of the candies would stimulate students' needs of using structure.

We also provided some examples of packing such as packing of pocket tissue, tea boxes and egg boxes. We hoped students would realize that they can find structures everywhere, and be more aware of it. We expected students would be able to use the structures of the packing to determine the number of objects in that packing by conceptual subitizing.

The candy activity would be followed by a classroom presentation in which each group would have to present their candy packing. In this presentation, students would explain the structures in their packing and how those structures help them determine the number of candies easily. From this presentation, we expected students would see different structures such as groups of 5 or groups of 10 and then they would be able to compare those structures and finally choose the best structure that is most effective in helping them counting.

Activity 1.1: the candy packing

The teacher started the lesson by telling a story about her experience in a store, buying candies. This story was used to develop the context of the lesson and to raise the problem of candy packing. As we have predicted, at first students used counting all strategy as they did not have the need to count by using other ways. To stimulate the students constructing structures, the teacher asked them to arrange the candies on the table so that the arrangement should help them doing a faster counting.

Students showed a good cooperative work as they discussed within their group about how the arrangement should be. Some groups used groups of 10, which indicated that students were already familiar with tens. Other groups used groups of 5, in one of the group that used groups of 5, we observed that since there are 4 students in the group, they divided the candies equally so that each student got 5 candies, and then they grouped the candies by 5. This implies that number of students in the group influenced how they worked.

In our observation, we found a group of students used groups of 10, they put the candies in 2 lines consisted of 10 candies each. When asked to explain how that structure help them counting, they said it was easy because 10 and 10 was 20. But they were still using counting all to find out that there were 10 candies in each line. This indicated that students have known groups of 10 and that adding tens was easy for them. However, it is not possible for students to subitize 10 candies when they are structured in a line.

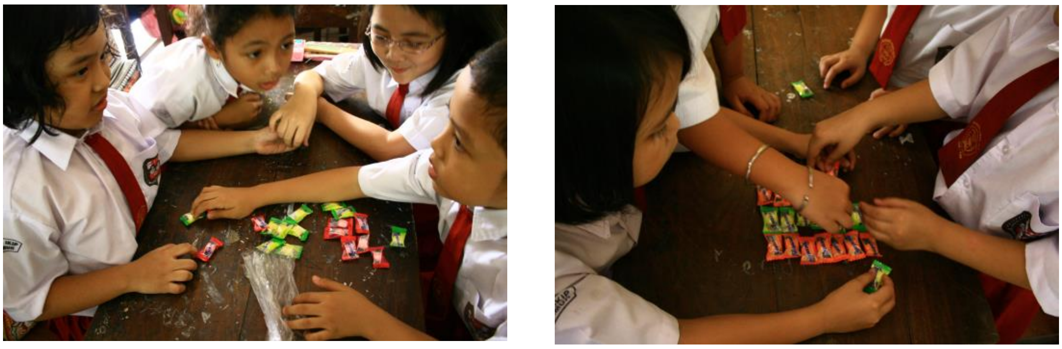


Figure 5.1: Students cooperative work

Activity 1. 2: Group presentation

Figure 5.2: candy arrangements

The group work was continued by group presentation. Each group was asked to make a drawing to represent their candy packing. This drawing would be used in the group presentation as the *model of* their candy packing. But since students' drawing were not too clear, the teacher improvised by asking the students to stick the candies in a paper.

During the presentation, the transition between concrete objects to abstract mathematics can be seen clearly. By using the candy arrangement, students can give an oral reasoning of their counting strategy. Group 1 was the first to present their work, they used groups of 10, but since they did not give a strong argument, we decided not to discuss their work in this report. The following fragments were chosen because we can see how students used the structures for counting.

<u>Group 2</u>

Ghea	:	The candies are divided into fives. (Pointing the candies with a ruler) 5 plus 5 is 10. This is another ten, so all together is 20.
Teacher	:	Look at the candies, how many are in this upper row?
Students	:	5.
Teacher	:	And how many are in the row below it?
Students	:	5.
Teacher	:	(Pointing to the candies). And in this row?
Students	:	5.
Teacher	:	And the last row?
Students	:	5.
Teacher	:	So, how do you count it? 5 plus 5 plus 5 plus 5. Who wants to ask?
Fariz	:	Why do the red and the orange candies have different length?
Teacher	:	Why the lengths different? The orange is shorter than the reds, Ghea?
Ghea	:	The orange candies are too close to one another, they are more crowded, that's why it looks shorter.
Teacher	:	Yes, they are too crowded. Come here Fariz, count the candies yourselves, are they four or five?
Fariz	:	(Steps forward and counts the candies).
Teacher	:	Who else wants to ask?
Indi	:	(Very weak voice) why did you divide them into fives?
Teacher	:	Indi asked why Ghea and her group divided the candy into fives. Because it's easy to count, isn't it Ghea?
Ghea	:	(Nod her head).

In this fragment, Ghea and her group used groups of 5. However, in the explanation, Ghea combined groups of 5 and groups of 10 together. First she said "5 plus 5 is 10". She added five candies in the first row, and five candies in the second row, and she got 10 candies. Next she said "there's another 10", this might imply that she recognized the same structures in the last 2 rows. Instead of repeating the addition

5 and 5, she shortened up by joining the last two groups of candies together and got 10. The groups of 5 have allowed her to do conceptual subitizing.

We noticed a critical moment when the teacher asked students the number of candies in each row. By doing this the teacher stressed out on the structure of the groups of 5. Recognizing structures is one of the important steps of bringing concrete objects to abstract mathematics.

Group 3

Safira	:	(Pointing to the candies, very weak voice) each line has 5 candies … (inaudible).
Teacher	:	Can you hear that?
Students	:	No.
Teacher	:	(Settling down the class) There are students who still walk around and talk. Can everybody please listen? Safira said that she and her group arranged the candies vertically. How many are in this line?
Students	:	5.
Teacher	:	(Pointing to the candies) this line?
Students	:	5.
Teacher	:	This line?
Students	:	5.
Teacher	:	And this line?
Students	:	5.
Teacher	:	So in total?
Students	:	20.
Teacher	:	(Pointing the candies) up to here are five candies, up to here?
Students	:	10.
Teacher	:	Up to here?
Students	:	15.
Teacher	:	And all of them?
Students	:	20.
Teacher	:	Is there any questions?

In this fragment, the classroom situation was not conducive, students were busy talking and they did not listen to Safira. To get students' attention back, the teacher evaluated students' performance. Students immediately calmed down and paid attention. This indicates that students needed to be reminded to stay calm most of the time. Students were still struggling to develop a classroom culture in which each member of the class has a responsibility to listen while their friend is explaining something in front of the class.

The teacher was still emphasizing on the groups of 5, she pointed out at the number of candies in each line. By doing this she showed the relation of the structures with the mathematical arguments.

<u>Group 6</u>

Gina	:	There are two colors, red and green. There are 10 greens and 10 reds, so five plus five equal ten. Ten plus ten is twenty, so in total, there are twenty candies.
Teacher	:	Gina's group made how many lines?
Students	:	2.
Teacher	:	Yes, 2. Ten and ten. Gina should have not explained about five plus five. It is ten and ten.
Fathur	:	Miss, I want to ask.
Teacher	:	Ok, Fathur want to ask, yes Fathur?
Fathur	:	Why they put the drawing in the same paper?
Teacher	:	Because they made big drawing so that it did not fit in the paper. So, I asked them to put the candies and the drawing in the same paper. Ok, Fathur?

Group 6 grouped the candies based on its color, red and green. Each color consists of 10 candies. It implies that the flavors or the colors of the candies have promoted students to use groups of 10. However Gina used groups of 5 in her argument. She might have been effected by the previous presenter who used groups of 5. The teacher immediately corrected Gina's answer and stressed out on the groups of 10. The teacher's intention was to synchronize the structure and the mathematical reasoning. This could have been an interesting moment if she had asked Gina why she used groups of 5 instead of groups of 10.

<u>Group 8</u>

Rara	:	In each line, there are 10 candies. (Pointing to the candies, dividing each line into two groups). In this line, there are five. In this line, there are five thus, in total there are 10. And then, here are also five, next to them, are five as well. So five plus five is ten, another five plus five is ten. Ten plus ten is twenty.
Teacher	:	Ok, give applause for Rara and her group. Now, I see some of you are busy doing your own thing. I want everybody's attention.
Students	:	Miss, Fariz wants to ask.
Teacher	:	Wait, I'll explain what Rara said. Rara said about five and five, but the candies are grouped into ten. Shortening up, how many are in the upper line?
Students	:	10.
Teacher	:	And in the lower line?
Students	:	10.
Teacher	:	So altogether?
Students	:	20.
Teacher	:	Is there any question?
Fariz	:	Why did you put the candies, red-orange-red-orange?
Students	:	So that it has a pattern
Rara	:	So that it looks nice
A Student	:	So that we can do counting by two
Students	:	Yes, that's right. We can do counting by two

There is a similarity in group 6 and group 8's presentation. Even though these students used groups of 10 where they made 2 lines of 10 candies, in her explanation Rara used groups of 5. It shows an inconsistency between using the structures and strategy of counting. Like Gina, Rara might have been influenced by the previous groups that using groups of 5. They could not instantly convince her classmates that there are 10 candies in one line since 10 is a big number, therefore they divided each line into smaller groups.

The teacher maintained emphasizing on the structure of groups of 10. She concentrated on the groups of 10 candies, since there are 2 groups of 10, thus altogether 20 candies. In this fragment, the teacher demonstrated the relation between the structures shown and the counting strategies.

After all groups have presented their work, the activity was continued with a whole class discussion. In this discussion, the teacher asked the students to choose one arrangement that allows them to count the easiest.

Teacher	:	Now, look at these arrangements you made. You see everything? From group 1 to group 9. Do you think which is the best arrangement that allows you to count easy? Raise your hand, and be quiet.
Fikri	:	Group 9.
Teacher	:	Group 9? Compare your works to the others. Don't say your work is the best just because you did it. Look at the other's work, are they better than yours?
Kasya	:	Group 1.
Teacher	:	Why Kasya?
Kasya	:	Because there are 10 candies in the first line, and 10 candies in the second line. So altogether is 20.
Teacher	:	Do you agree with Kasya? Do you have the same reason with Kasya?
Students	:	Yes.
Teacher	:	Kasya said, that here in the first line, we have ten candies, and in the second line we have 10 candies, so in total there are 20 candies. (writing a formal mathematical sentence) 10 + 10 = 20. Which group has the same arrangement?
Students	:	Group 8.
Teacher	:	Group 8 has a nice arrangement. What else?
Students	:	Group 6.
Teacher	:	Yes, group 6 also has 10 candies in each line. However the arrangement is a little bit wavy. Anything else?
Students	:	Group 4.
Teacher	:	Group for is the same too. But the red candies and the green candies are not equally distributed. It might seem that there are more greens than reds. So next time, will you make it better? Does anyone have different opinion?
Dinda	:	I think group of 5 is easy.
Teacher	:	Group of 5, so which group do you choose?
Dinda	:	Group 5.
Teacher	:	So you think this is easy. 5 and 5 and 5 and 5. But which arrangement that you think is easier than this?
Students	:	That one, group 2 and group 3.
Teacher	:	So there are two different ways, grouping by 10 and grouping by 5. (Approaching the work of group 2 and group 3, writing the mathematical

		sentence) 5 + 5 + 5 + 5 = 20. Is there any other arrangements?
Students	:	7.
Teacher	:	This one, but this was because the paper was not long enough. So basically they grouped the candies into tens. So we don't have other arrangement, only grouping by five and grouping by ten.
Teacher	:	In what packaging do you see this kind of arrangement?
Students	:	Calculator.
Teacher	:	What packaging?
Students	:	Biscuit.
Teacher	:	I have right here. What is this?
Students	:	Tissue.
Teacher	:	Can you tell how many are they (showing only the front view)
Students	:	5, 10.
Teacher	:	Turning the tissue) how many?
Students	:	10.
Teacher	:	Why?
Students	:	Because there are 5 in front and 5 in the back.
Teacher	:	Gina, come and explain why they are 10.
Gina	:	There are 5 in this line. And there's another line which also has 5. So all of them are 10.
Teacher	:	Gina said here are 5. Is that true?
		Yes
Teacher	:	1, 2, 3, 4, 5 but we have another line. We have 5 more, so altogether we have 10. So this is the pattern of five. In which group did you see such pattern?
Students	:	Group 5
Teacher	:	It's like group 2, group 3 and group 5. I have more in here. (showing a tea box) How many are these?
Students	:	6.
Teacher	:	Why 6? In the front line, how many?
		3.
Teacher	:	We have two lines, so in total there are 6. Now, what do I have here? What packaging is this?
Students	:	Eggs.
Teacher	:	How many eggs?
		10.
Teacher	:	(Approaching Kasya) How did you figure it out?
Kasya	:	5 and 5. 10.
Teacher	:	How did you count it?
Kasya	:	5 plus 5 is 10.
Teacher	:	Ok good. You see that these eggs are arranged nicely, so we don't need to count one by one, one, two, three, …, ten. Just look at the lines. How many are in the first line?
Students	:	5.
Teacher	:	The second line?
Students	:	5.
Teacher	:	Yes, isn't it easy?
Students	:	Yes.

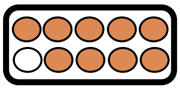
In our HLT, we wanted to achieve a classroom agreement of which structure is the best. However, we observed that during the discussion each student has different opinion of the best arrangement, some preferred groups of 10, and some other preferred groups of 5 and no agreement was made. The best arrangement would be the one that students are convenient to work with, and it differs to every student. This indicated that this class hasn't developed a socio mathematical norm in which students work together to determine the best and fastest way of counting. Even though the classroom atmosphere was very open in a sense that students were allowed to use their own strategy. However, students were not used to give a strong and convincing argument.

In the last part of the discussion, the teacher showed some products that were easy to find in store such as a pack of pocket tissue, tea boxes and egg boxes. Students were able to reason orally about determining the number of objects by conceptual subitizing. Students were able to count using the structure (i.e. there are six tea boxes, because there are three in the front line and three more in the back line), counting all did not appear during this discussion,. This indicated that the previous activity, namely the candy packing has built students' awareness of structure.

Activity 1. 3: Flash card game

The lesson was closed by playing flash card game. In this game, students only had a few second to determine the number of objects shown in the cards. The time limitation promoted students to employ the structures for doing conceptual subitizing. During this game, we found that students were already familiar with finger structures and dice structures, and they also did not find any difficulties working with egg box structure.

Teacher : (Showing a card)

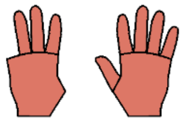


Students : (Raising their hand)
Teacher : Laras?
Laras : 11.
Teacher : 11? The others? (pause) Ihsan?
Ihsan : 9.
Teacher : Why?
Ihsan : (very weak voice).
Teacher : One is missing, right? Let's see together (showing the card)
How many altogether? (pause).
Students 10.
Teacher : One is missing.
How can you tell that there are 10 so easy? (pause)
Because here are 5 and here are 5.

It is not clear how Laras said that there were 11 eggs. The teacher did not ask her further, but we may interpret that Laras might have seen 6 eggs in one row, she recognized one was missing, thus she came with an answer 11.

In this fragment, Ihsan used groups of 10 structure, he subtracted 1 missing egg from the whole collection of 10 eggs. He argued that one egg is missing from the box so that only 9 are left. We did not see students argued with group of 5; 5 + 4 = 9.

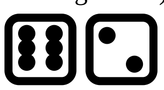
Teacher : (Showing a card)



Students : (Raising their hand).
Teacher : Tasya?
Tasya : 8.
Teacher : Why Tasya?
Tasya : 4 + 4 is 8.
Teacher : That's not what the card shows. Dini?
Dini : 9.
Teacher : You did not see it correctly. Ais?
Ais : 10.
Teacher : 10? Salma?
Salma : 8.
Teacher : Why?
Salma : Because 3 plus 5 is 8.
Teacher : Is it 3 plus 5? Let's prove it (showing the cards)
Students : Yaa.
Teacher : Did you need to count the full hand?
Students : No.
Teacher : No need to count them all over again, because we now that our hand has 5 fingers.

When the teacher showed a card, Tasya knew that there were 8 fingers, but her argument did not fit the drawing. Tasya said that 4 plus 4 was 8, while the card has shown 5 fingers and 3 fingers. Tasya might have been able to recognize finger structures, but she had forgotten how the structures were made of. Dini and Ais did not look at the card correctly; they probably did not pay full attention when the card was shown. Salma gave a good mathematical reasoning; she could tell that there were 8 fingers because 3 plus 5 is 8. This indicates that she has understood the fingers structures.

Teacher : (Showing a card)



Students : (Raising their hand).
Teacher : Lifi?
Lifi : 8.
Teacher : Why .
Lifi : 6 + 2 .
Teacher : 6 + 2. How can you tell that there are 6 so quickly?

Lifi	:	3 + 3.
Teacher	:	Lifi said, she did not count I, 2, 3, 4, 5, 6. That takes so much time. But there are 3 and 3. So 3 + 3 + 2 is how many?
Students	:	8.

Lifi could tell that the card showed 8 because of 6 + 2, this indicates that she knows dice structure very well. This indication is also supported when she argued that she recognized 6 dice by 3 plus 3. The teacher kept encouraging the students to use structures all the time. She showed that by using structures, students can do a faster counting rather than counting all.

Students played this game with more cards, throughout the game we found that students were very enthusiastic that they raised their hand as quick as possible to get the turn to answer. They showed disappointment when the teacher did not choose them to answer. In our next HLT, instead of telling the number orally, we will ask students to raise a number card to show the number represented in the card. In this way, all students get the same opportunity to answer.

We also found that students did not have any difficulties in recognizing finger structures, dice structures or egg box structures. However, in this activity, students haven't fully explored the egg box structures. They use groups of 10 argument most of the time, for instance, there are 7 eggs because 3 eggs are missing. In this argument, they used groups of 10 structure because they can recognize 10 eggs in a full box. They haven't used the groups of 5 reasoning, since 7 eggs can also be 5 eggs in the first line and 2 eggs in the second line.

In short, throughout this lesson students have become aware of the need of using structure in doing a better counting. Instead of doing counting all, students can do counting by grouping. Counting by grouping allowed students to determine the quantity of a collection of objects by conceptual subitizing. When counting, students can understand the importance of keeping object structured. This activity has also provided a bridge for students do develop their thinking process, going from concrete object to abstract mathematics.

We observed that students have been able to use the structures in their counting strategy. Students have been able to determine the number of object by using the structure in their counting strategy. They also have been able to communicate their thinking orally, but not the written formal mathematics.

5.2. Lesson 2: Double structure (08 August 2008)

Up to this moment, students have become more familiar with structures, and now our goal was to explore the double structure and double sums. In our HLT, we introduced the double structure through a song called "*Satu ditambah satu*"/"One plus one" or the double song.

The teacher would start the lesson by asking students to sing the double song together. We expected that students would be able to memorize some double sums through singing the song. While singing, students would be asked to make a group consisted of a particular number of persons according to the lyric of the song. For example when singing "*satu ditambah satu* (one plus one)" they would make a group of two persons. We conjectured that there would be many physical activities as students would walk around for their groups. These physical activities might ruin the structures in the group; therefore we anticipated it by asking the teacher to always encourage the students to conserve the structures in the group.

This activity would be followed by a coloring activity. In this activity, students worked on a worksheet in which they would have an empty candy packing to be filled out. They would be asked to fill out the candy packing by coloring the picture. By the way students color the candy packing, we would be able to see if they use double structure or not.

Activity 2.1: Singing the double song

The teacher started the activity by discussing the advantages of using structures in a group of objects to enable students count the number of objects easily. She gave some examples by using students' seat arrangement. In Indonesian classroom, students sit in 4 columns, each column consist of 10 students, and every 2 students sit together behind a table (figure 5.3). These structures were used by the teacher by asking how many students were sitting in each column.

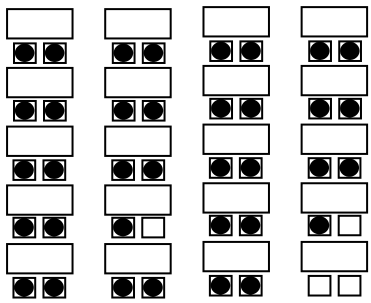


Figure 5.3: students' seating arrangement.

Students could see immediately that there were 10 students in the first column and in the third column by subitizing. Wira grouped the students by the chair they were sitting on, he argued that there were 5 students in the right side, and 5 students in the left side, so altogether are 10 students. Kasya could tell immediately that there were 9 students in the second column because she saw that there were 5 students in the right side and 4 students in the left side. The teacher asked other students if they have a different way of knowing 9 students in the second column. Gina reasoned that there were 9 students because there were 10 chairs in the second column, and 1 chair is empty. Here, we observed that students have been able to recognize groups of 5 and groups of 10 structures in students' seat. Students used those groups to determine how many students were sitting in a column by conceptual subitizing. They could see that each column was composed from 2 groups of 5.

During the song activity, we observed that all students were very enthusiastic. Like we had predicted in our HLT, students' physical movement destroyed the structures in their group. The teacher encouraged all students to preserve the structures in the group so that they would stand in a structured configuration. Students were encouraged to always make a structured configuration so that it is to determine the number of the students in that group. As the sums got bigger, some students did not make a group because there are not enough people in the class. Students who did not get a group, played a role as the judges, they had to ensure that each group consists of the correct number of students.

When singing "*delapan ditambah delapan* (eight plus eight)" students formed 2 groups of 16. Since there were only 37 students in the class, 5 students remained without a group. These 5 students played a role as the judges; they observed the two groups and determined the number of students in each group. This was an example of the teacher's improvisation; she has a nice technique to engage all students in the activity. Group A made a 3 by 5 configuration with one extra student, and group B made 2 by 8 configuration.

Group A Group B

Figure 5.4 : Students group configuration

The following fragment showed the interaction during the activity.

Bintang	:	3, 3, 3 , add all of them.
Teacher	:	Ok Class, please listen to Bintang.
Bintang	:	3 + 3 is 6, 6 plus 3 is …(pause) 9, plus 3 is 12 plus 3 is 15 plus 1 is 16.
Teacher	:	Yes, 3 plus 3 plus 3 plus 3 plus 3 plus 1 more. So 15 plus 1 is 16. This is why you should keep your position in ordered lines, so that it can be countable easily. Ok, now, Bintang, can you tell how many are in the other group.
Bintang	:	(counting).
Teacher	:	At the beginning, Bintang used counting one by one, could you explain why, Bintang?
Bintang	:	Because they were not standing in order.
Teacher	:	Ya, exactly. You see, if you're standing unordered it will be very difficult to count. Andini, could you step aside and join the judges please? Now, the 2 by 8 group (group B), can you tell how many students in group A now?
Students	:	(counting).
Fathur	:	15.
Teacher	:	How did you get that?
Fathur	:	There are 5, and 5, and 5, so it's 15.
Dinda	:	I know a different way.
Teacher	:	Yes Dinda?
Dinda	:	I know that there were 16, and you called one out, so 16 minus 1 is 15.
Teacher	:	Very good Dinda. Now I'm calling Vicka and Adiza to step aside. How many are in the group now?
Wira	:	13 because 3 were taken out.
Teacher	:	Haura, tell how you count them.
Haura	:	(pause).
Teacher	:	Did you do counting one by one?
Haura	:	(pause).
Teacher	:	Ok, Haure doesn't want to answer. Salma how did you do it?
Salma	:	Because 5 over there, and 5 over there, and 3 in the middle.
Teacher	:	Ya. Very good.

Instead of using groups of 5, Bintang used groups of 3, this probably caused by his point of view. During this activity, Bintang were facing the group in a way that he could immediately see there are 3 students in a line. He did not look for other option, (i.e., there are also 5 students standing in a line). Bintang could easily recognize 3 students by subitizing, and used the groups of 3 for his counting strategy. However, he still needs to improve his counting skills as he was still doing counting on when determining 6 + 3. Here, we observed that Bintang has been able to recognize the structures, that is when he could divide a group into smaller groups and add the quantity of the smaller groups to get the amount of the initial group. However, he did not know how to add the numbers which might have been caused by his lacking of understanding the basic addition up to 10.

The teacher emphasized on the importance of structuring for doing a better counting. She gave a contrast comparison between the unstructured group and the structured group that when the group is structured, it is easier to count. When the teacher called some students to step aside from the group, the student still did not have any difficulties finding out the number of students left in the group. This was because the structures were preserved. Moreover, students could give various reasons, for example, Fathur agued there were 15 students because 5 plus 5 plus 5, while Dinda argued there were 15 because 16 taken away 1.

However, few students still did not get the idea of structuring, Haura was one of them. She felt insecure when the teacher asked her to tell the number of students in group A. It might have been caused by her incompetence to use the structure of the group, since she was still using counting on and could not do counting by grouping. This could have been a critical moment for Haura to learn counting by grouping if the teacher had given her an opportunity to try it out. In our next HLT we should consider the learning of low achiever students like Haura by showing her other strategies which are faster and encouraging her to use those strategies while counting.

Based on our observations, most students were fully engaged and they were working cooperatively in their group. Each student played an important role in their group since they had the same responsibility to keep their group nicely ordered. Students who did not get a group also played an important role by being the judge so that they did not feel left behind.

During this activity, counting by grouping has been chosen by most of the students, even though few of them were still counting all. Other strategy that was used by the students is counting by subtraction. Since they knew the initial number of students, when some students were called out from the group, they simply used subtraction to determine the number of student left. This indicates that students have understood the concept of subtraction that is when some objects are taken away from a group of objects.

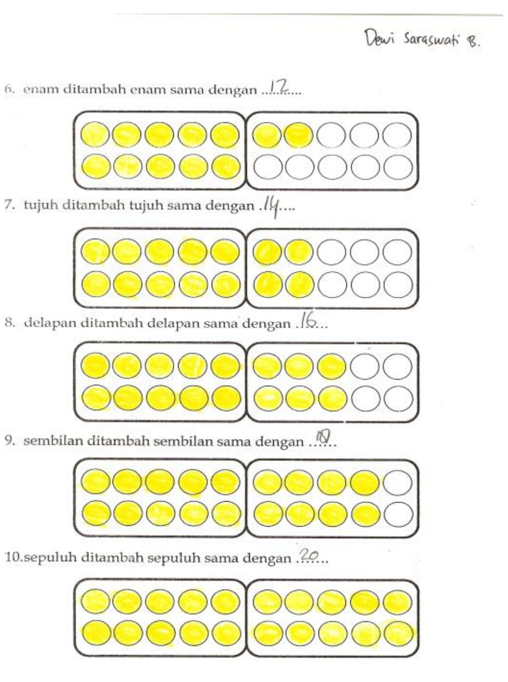
We also found differences in students' ability of structuring. There were students who did not aware of the structures, thus they were unable to use the structures for further counting. These students kept on relying on counting all strategy. There were also students who could recognize the structures, and use it in counting. However, they inability of basic additions disallowed them to do counting by grouping; therefore they still used counting all. Lastly, we found some students who were able to recognize structures and use it in counting by grouping or conceptual subitizing.

Activity 2.2: Structuring by coloring

In this activity students worked on a worksheet in which they were asked to filled out some candies in an empty packing by coloring it. The numbers of candies given were all double sums. We conjectured that students would construct double structure by making 2 identical groups in each packing. Next, students would be asked to tell the number of candies in a packing and give a reasoning of their thinking. We expected students would use double structures in their thinking process.

Throughout the coloring activity, we observed that students produced written works which indicated that the double song did not immediately promote students to memorize the double sums. The observation showed that while working on the double sums, the low achiever students still use a counting strategy. This finding has raised some question, i.e., why students did not use the song as a way to memorize double sums and what caused low achiever students to rely too much on counting on strategy? The double sums in the song were probably too abstract for the low achiever students or for them the lyrics of the song were just a regular words instead of representing numbers.

In the observation, we also found that students showed a good understanding of structuring which was seen in the work they made. The coloring activity showed that most of the students made a double structure. However, we still found few students who did not use a double structure.



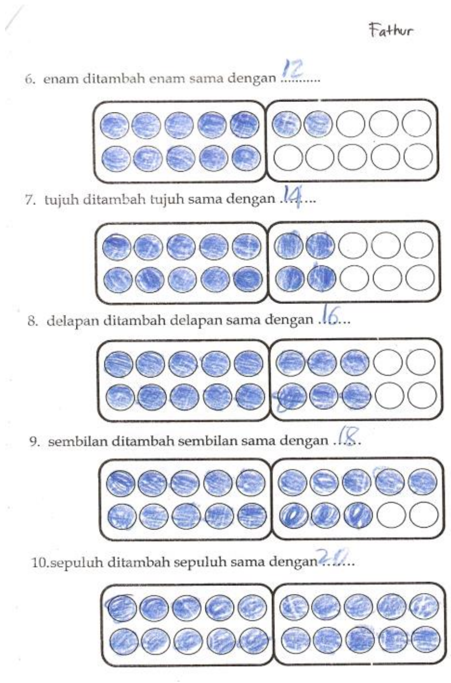


Figure 5.5 : Students' coloring

In the next task, students were asked to determine the number shown in the picture of double structures. We found that only few students use double arguments. There are some strategies used by the students besides doubling, i.e., counting all, counting on, addition and subtraction.

Prawira used subtraction; he looked at the empty candies and subtracted them from the whole number of candies. Janet on the other hand used double when determining 4 and 10 candies, but when worked on 14 candies she used groups of 10. This might caused by the structure given in the picture. The double structures were shown clearly for 4 and 10 candies. Even though the double structure for the 14 candies was visible, but the groups of 10 structure was more visible for Janet.

From Dewi written works, she did not explain the mathematical reasoning. We assumed she used counting all or counting on when working on this worksheet. This indicated that she has not grasped the double structure and moreover, the coloring activity might have been meaningless for her. Fathur used double strategy, moreover he could write the formal addition. This indicates that he has conceived the idea of double structures through the singing and coloring activities.

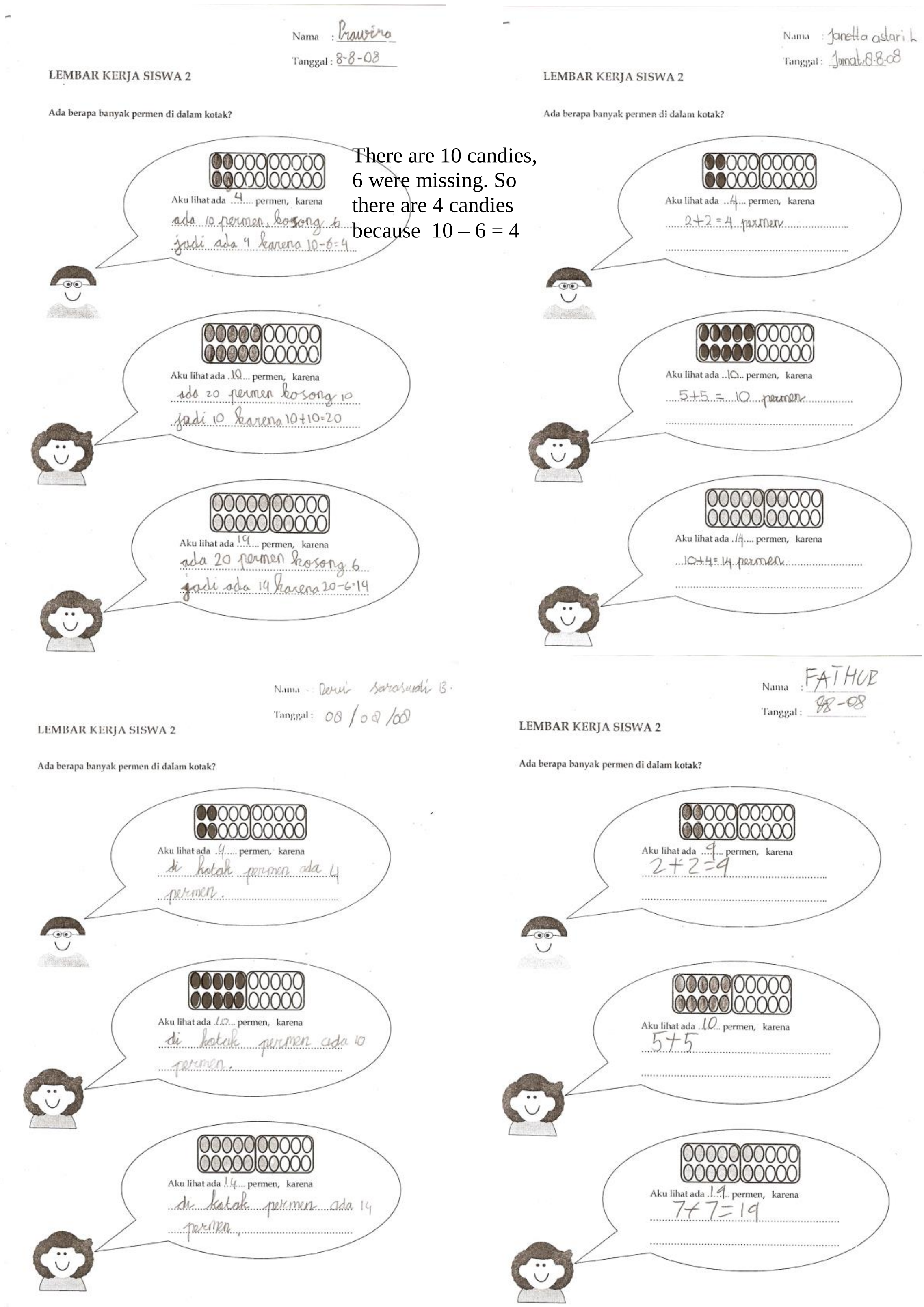


Figure 5.6: Students' counting strategy

Summing up, we concluded that the singing and grouping activity could be a nice start for introducing double structures and double sums. Students were actively engaged in making the grouping. However, we recognized a big jump from the singing activity to the worksheet activity. Students did not automatically use the double sums in the song to help them determine some double sums problems. Even though singing is a fun activity for students, but we discovered that the mathematics in it was still too abstract for students as many of them still could not relate the lyrics of the song to the mathematical objects.

We also found that the grouping while singing did not help students conceive the idea of double structure. The reason for this might because students were a part of a group. They played a role as a member in the group which made it difficult for them to participate and observe the structures at the same time. Thus, they could not see the construction of the structures clearly. This finding gave us input for the improvement in the next HLT that the singing should be followed by a more hands-on activity so that students can experience the double structure.

The coloring activity might have given more impact on students if it had been followed by a discussion. For the next HLT, we propose that students' different structures can be brought to a classroom discussion in which students would compare each structure. The teacher will guide the discussion so that students are exposed to double structures and how to use it in telling the quantity of a group of objects.

5.3. Lesson 3: Addition up to 20 with double strategy (14 August 2008)

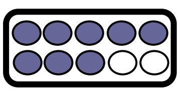
Since this lesson was the continuity of the previous one, we used our observation of the previous lesson to determine the starting point of this lesson. The observation showed that the previous activities have promoted students to use structures informally when counting, or in other words, students were able to explain orally how the structures was used in their counting, but not in a formal mathematical expression. Therefore in this activity we repeated exploring on the structures and reinforced students' understanding by guiding them to the formal mathematics.

First, students would repeat an activity on the egg box structures. They would be asked to tell the number of the eggs and reason about it. We conjectured that students would be able to use the groups of 5 structure and groups of 10 structure when reasoning about how many eggs were in the box. Next, students would be guided to write down their reason in a formal mathematics expression. For example, if a student can reason that there are 6 eggs because 3 are in the upper row and 3 are in the lower row, then he or she should be able to write: 6 = 3 + 3.

The second activity was the “The sums I know” worksheet. The aim of this activity was to stimulate students to construct a number relation, for example, 15 is 14 and 1 more. Then they could use the number relation to solve some addition up to 20 problems, especially the almost double sums. For example 7 + 8 = 15 because 7 + 7 = 14 and 7 + 8 = 7 + 7 + 1. The third activity was exploring the math rack. Students would learn the groups of 5, groups of 10 and double structures in a math rack. We conjectured that through trying out working with the rack, they would be able to discover the structures in it.

Activity 3.1: Reviewing the egg box structure: from informal oral reasoning to a formal written mathematical expression.

The teacher started the lesson by showing an egg box and a card of egg box structures. Students were asked to tell the number represented in the card. The following fragment showed students’ learning process.

Teacher	:	(Showing an egg box with 10 eggs) Who can tell how many eggs in here?
Students	:	Me.
Teacher	:	Farrel, now I’m asking Farrel. Others please listen. How many Farrel?
Farrel	:	10.
Teacher	:	Why? Could you explain how you count?
Farrel	:	Because 5 + 5.
Teacher	:	Who agrees with Farrel?
Students	:	(Raising their hands).
Teacher	:	Yes, 10 eggs because 5 here and 5 here. Now I’m going to take away some of the egg. Let see if you can tell how many eggs in the box. (taking away 3 eggs).
Students	:	(Raising their hands and shouting) 7, 8.
Teacher	:	Please be quiet. I’ll ask Dinda. How many Dinda?
Dinda	:	8.
Teacher	:	Who agrees with Dinda?
Students	:	Noo.
Teacher	:	I want to know why Dinda said 8 eggs.
Dinda	:	Because 10 take away 3. (pause… looking at the egg box)
Teacher	:	10 eggs and taken away 3 is 8?
Dinda	:	No, it’s 7.
Teacher	:	Do you agree now with Dinda?
Students	:	Yes.
Teacher	:	Now, I’ll use these cards. Can you tell how many eggs are shown? (Showing a card). 
Naga	:	8.
Teacher	:	Can you write it mathematically?

Naga	:	(Writing on the white board). 8 = 5 + 3.
Teacher	:	Is this correct? Who has the same answer as Naga?
Students	:	(Raising their hands) I do.
Teacher	:	Who has different way?
Dinda	:	I know a different way.
Teacher	:	Ok, Dinda come forward and write it down.
Dinda	:	(Step forward).
Teacher	:	You have the same answer as Naga, right?
Dinda	:	(Thinking)… Yes. (Writing on the white board). 8 = 10 + 2.
Students	:	(Spontaneously react) nooooo.
Teacher	:	Dinda, is it really 10 + 2?
Dinda	:	No. (Correcting her answer). 8 = 10 – 2.
Teacher	:	Is this correct now?
Students	:	Yes.

In this fragment, the transition from concrete objects to schematized objects to formal mathematics can be seen clearly. At first the teacher used a real egg box and asked students to tell and reason about the number of eggs in the box. Students were very enthusiastic in giving the answer. When having 10 eggs students could immediately recognize the groups of 5. But when given 7 eggs, Dinda used the groups of 10 structure and subtracted the missing eggs from the 10 eggs. At first she said that from 10 taken away 3 was 8. But then she counted the eggs one by one and found that there were 7 eggs. This indicates that Dinda has not mastered basic additions and subtractions up to 10. She recognized the groups of 10 structure but she could not do the subtraction correctly, therefore she still needed to count all eggs one by one.

After having the real egg box, the teacher changed to using a card which was also a schematized egg box structures. Here the card served as the *model of* the egg box. Naga used group of 5 and he could write down the formal mathematics of the addition, i.e., 8 = 5 + 3. In this stage, the card was no longer a *model of*, but it has become a *model for* the groups of 5 structure. Naga used the structure to reason that 8 = 5 + 3.

Dinda used the groups of 10 structure. Again, she showed that she has recognized the structure but she did not know how to use it in counting. When writing 8 = 10 + 2, she knew that there were 10 eggs in total and 2 eggs were missing, but she did not know how those numbers relate to each other. She got an immediate response from the class that her answer was wrong, and then she changed it into 8 = 10 – 2. Dinda seemed haven't fully grasped the basic additions and subtractions up to 10.

We also observed the socio norms that was practiced by the teacher. She asked all students to listen to Farrel when he was telling his strategy. Through this norm, students learned to learn from others by listening and sharing their thinking. We also found another socio norm in this class, when Dinda made mistakes, the class always gave an immediate feed back without discouraging her. This was a supportive norm for Dinda since she needed to know her mistakes and correct them immediately.

Based on our observations, we concluded that this activity has shown the transition from *model of* to *model for*. At first the card was used to replace the egg box, here the card served as a *model of* the egg box. Gradually, students no longer related the card to the egg box, but used the structure it to give a mathematical reasoning. Here, the card has transformed into a *model for* the groups of 5, groups of 10, or double structures.

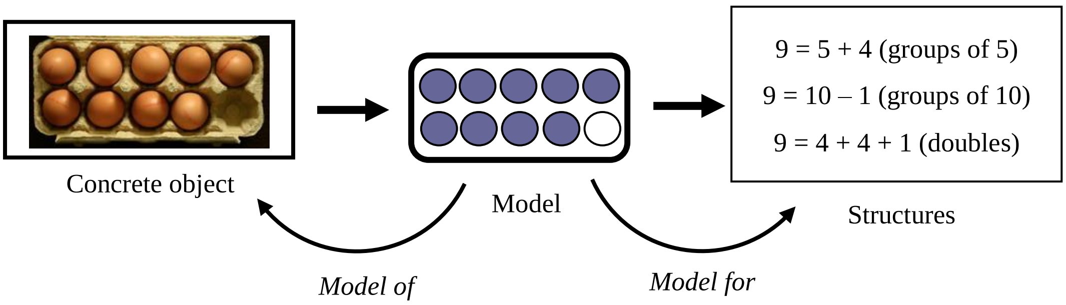


Figure 5.7: From *model of* to *model for*

Activity 3.2: "The Sum I know" Worksheet

The main activity in this lesson was the "The Sum I Know" activity. Teacher started this activity by first asking the students to sing the double song, and while singing, they showed the addition by using their fingers. The singing was meant to remind the students of the double sums before students worked on the worksheet. In this worksheet, students were given 5 minutes to circle all the sums they knew by heart so that we could minimize the possibility of students to use counting. We conjectured students would circle the first row or the first column and then they move to the second, etc. We also expected students to circle some double sums since they have been doing some double through singing in the previous activities.

After 5 minutes student would be asked to stop circling and begin to write the result of the circled addition. By writing the result, we hoped some students would discover the number pattern, such as every time they move one step to the right, the addition gets one bigger. Then the teacher would use this discovery to open a discussion about addition framework. In this discussion, students would be guided to use the sums they know to solve other sums. For example, if students have known 8 + 8 = 16, they can easily know that 8 + 9 = 17 because it's located next to 8 + 8, or 8 + 9 = 8 + 8 + 1 = 16 + 1 = 17.

The observation showed that most of the students started circling from the first row, and then they moved to the second row, etc. They also circled the last row and the last column. Only few of the students circled the double sums. This finding raised a question, why didn't students recall the double song when working on this worksheet?

After 5 minutes, students stopped working on the worksheet, but they did not continue writing the result of the circled sums. The teacher did not ask them to do so, instead she immediately moved to the white board and discussed the double sums in the table of addition. She asked students to sing the double song again, and while singing, she circled some additions sung in the song. After all the double sums have been circled, she asked the students to tell the result of those double sums. Students could answer it and made a series of numbers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20. A student discovered that those numbers showed jumps of 2 pattern, and other students said that they were even numbers. This discovery should have been a major learning moment for students if it had been followed by a deeper discussion. The jumps of 2 pattern can be used as a strategy of solving double sums. For example, if students have known that 8 + 8 = 16, then finding 9 + 9 would be easily done by 8 + 8 + 2 = 16 + 2 = 18. This could be an important addition in our next HLT.

In the following fragment, students learned how to use double sums to solve almost double problems.

Teacher	:	Now that we have known these double sums, we want to know other sums. Wira, what's the result of this sum? (pointing at 6 + 7).
Wira	:	(inaudible).
Teacher	:	Wira said how to get 7 into 10. Who has different way? We can use this table.
Faras	:	Over there 6 + 6 = 12. plus 1 more is 13, thus 6 + 7 = 13.
Teacher	:	Yes, I'll repeat Faras's and Wira's answer. Wira kept the 7 and make it into 10 by adding 3 more from the 6. So now there only 3 left. And 10 + 3 is 13. But Faras used a different way. He looked at this (pointing 6 + 6). 6 + 7 is right beside 6 + 6. So the result is 6 + 6 + 1. We knew 6 + 6 right? It is 12. 6 + 7 would be 12 + 1 = 13. Let's try again. Kasya what's the result of this? (pointing at 7 + 8).
Kasya	:	15.
Teacher	:	How come? Could you tell how you got that?.

Kasya	:	7 + 7 + 1.
Teacher	:	Yes, right. 7 + 7 + 1. Are you ready for another one? (pointing at 8 + 7).
Ihsan	:	15.
Teacher	:	Why?
Ihsan	:	8 + 8 = 16. We take one away 16 – 1 = 15.
Teacher	:	Oke. Very good. Who has different way?
Wira	:	8 + 7 is the same as 7 + 8.
Teacher	:	Yes, that's also true. We have found 7 + 8 = 15, so 8 + 7 is also 15 Is there any other way?
Faras	:	I see the 7 + 7, 8 + 7 is right below it, so 8 + 7 = 7 + 7 + 1 = 15.
Teacher	:	Yes, Faras looked at the 7 + 7, since 8 + 7 is located one box below 7 + 7, so you just need to add one more. 8 + 7 = 7 + 7 + 1 = 15.

Wira used decomposition to 10 strategy. Clearly he is used to do that and the double sum was not a natural way of solving addition up to 20 problems for him. Faras's performance during the activity was very impressive. He used the double sums to help him solve almost double sums. His strategy was exposed among the other students and it has stimulated other students to use the same strategy. Kasya solved 7 + 8 by doing 7 + 7 + 1 while Ihsan solved it by doing 8 + 8 – 1. This indicated that double strategy gave a flexibility for students, they can choose the double sums they know and then they can add 1 to it or subtract 1 from it to solve an almost double problem.

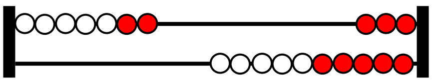
Throughout this activity, the teacher dominated the learning process. Students did not participate actively during the discussion. They tended to be a passive listener when the teacher was showing the strategy of solving almost double sums therefore they did not discover the number pattern by themselves. The reason for this might have been because students were not actively engaged in the activity since they only looked at the white board and listened to the teacher. In our next HLT, a discussion among the students might stimulate students to participate actively.

We missed the critical learning moment that we have expected in the HLT that it when students discovered the number relations concept through the worksheet. Students only did half of the "The sum I Know" worksheet. They only circled the sums they know, but did not write the result. It was very unfortunate because in our HLT, writing down the result could lead to the discovery of number patterns, and that did not happen in this classroom. In our next HLT, students must do the "the sum I know" worksheet completely.

Activity 3.3: Exploring the math rack

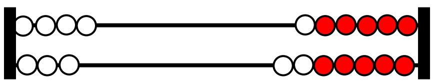
The last activity of the lesson was exploring the structures in a math rack. Teacher distributed the math racks, every two students got one math rack, and she had one big math rack in front of the class. The teacher asked the students to show a number by using the math rack. We conjectured that students would use different strategies to represent a number. Some students might use counting all by moving the beads one by one, some might be able to recognize the groups of 5 and used it for represent a number. We also conjectured students would have different representation of a number.

When asked to show *seven*, students had different way of representing *seven*. Naga was asked to do it in front of the class by using a big math rack. He moved 7 balls in the bottom bar of the math rack together. He explained, there were 7 balls because there are 5 orange balls and 2 white balls. This indicated that Naga could recognize the groups of 5 structure in a math rack and use it to represent 7. Bintang and Faras used counting one by one; they moved the beads one by one until they have 7 beads. Ryan and Raihan used almost double, they put together 4 whites in the upper bar and 3 whites in the bottom bar, and argued that 7 is 4 + 3.



7 = 5 + 2

Figure 5.8.a: Representation of seven using group of 5 structure

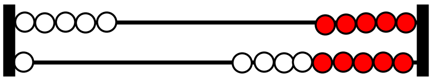


7 = 4 + 3

Figure 5.8.b: Representation of seven

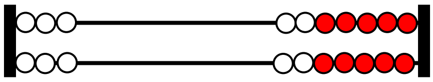
Figure 5.8: Representation of 7

Next, students were asked to show a *nine*. Bintang and Faraz moved 5 beads in the upper bar and 4 beads in the bottom bar. Faraz was the one who got the idea of using this representation, while Bintang was still thinking. We assumed that Bintang was counting 5 plus 4, and then he agreed that they were 9. When the teacher asked students to show a *six*, Faras still used group of 5. He put together 5 whites in the upper bar and 1 white in the lower bar. Bintang was looking away for a few seconds, and when he got his attention back, Faras has already represented 6 on the rack. Tasya moved 3 beads on the upper bar and 3 beads on the bottom bar. The teacher explained about the double structure that Tasya used, that is 3 plus 3.



6 = 5 + 1

Figure 5.9.a: Representation of six using group of 5 structure

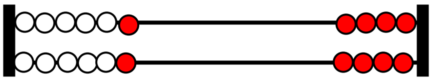


6 = 3 + 3

Figure 5.9.b: Representation of six using double structure

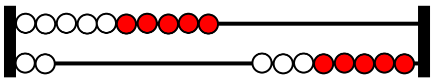
Figure 5.9: Representation of 6

Students tried another number, this time it was *twelve.* Bintang and Faras used double structure, 6 and 6, while Ezar and Ihsan used group of 10. They put all 10 beads in the upper bar and 2 more beads in the bottom bar.



12 = 6 + 6

Figure 5.10.a: Representation of twelve using double structure



12 = 10 + 2

Figure 5.10.b: Representation of twelve using group of 10 structure

Figure 5.10: Representation of 12

In this activity, students tried out different numbers using a math rack. Based on our observations, we conclude that some students have been able to use the double structure in the math rack, especially when it was used to represent even numbers such as 6, 8, 12, etc. Some students also used groups of 5 structure, especially to represent numbers less than 10. However, we also noticed that some students were still counting all by moving each beads one by one until they got the expected number. These students still haven't grasped the structures in the math rack. The teacher has given help by asking questions like "how many beads are in the upper bar, how many red beads" etc. These questions were aimed to direct students' attention on the structures of the math rack.

Finally, we draw some conclusions about students learning process in this lesson. In the first activity, the egg box structures were proven to be powerful to help students move from concrete objects to formal mathematics. First, students can reason the number of eggs by using the groups of 5 structure, groups of 10 structure, or double structure and then they put their reasoning in a formal mathematical expression. The cards as a schematized box served as a *model of* the egg box, which then transformed into a *model for* the groups of 5 and groups of 10 structures.

However, we observed a big jump between the previous activity (i.e., the candy packing) and the egg box activity as teacher did not make an explicit relation between these two activities. The egg box could have been used as a *model of* the candy packing students have made in the previous activity. We will make this relation clearer in our next HLT so that students can see the connection between the structures, the models, and the formal mathematics.

"The sum I know" activity did not happen as we have expected in our HLT. It might have been caused by the absence of students' discussion. Based on the observation, we could see that this classroom has not maximized the using students' discussion as an opportunity for students to learn from others.

In the last activity, we have seen that some students have discovered a fast way of representing numbers by using a math rack, they used groups of 5, groups of 10 or double structure. However, there were also students who still haven't understood the structures as they were seen using counting all. These students might need more time to explore the math rack.

5.4. Lesson 4: Decomposition to 10 strategy (15 Augusts 2008)

This lesson would focus on the friends of 10 and the decomposition to 10 strategy. We made a description of steps need to be done in performing the decomposition to 10 strategy. We set up 4 steps of doing decomposition to 10 strategy (chapter 2). In order to help students understand the strategy, the egg box structures can be used to underpin students learning of step 1 and 2. Students would learn about the number pairs that make 10 or the friends of 10 through the egg box structures. We conjectured the egg box would allow students to recognize the pair of a given number. For example, when there were 2 eggs in the box, students could easily determine the number of the missing eggs that is 8. This structure would stimulate students to grasp the friends of 10.

Next, for step 3 and 4, we conjectured the math rack would help students to perform the decomposition of the other addend. 5 is being decomposed in to 2 and 3 and not into other numbers such as 1 and 4 because 2 is the friend of 8. After students have understood this concept, we expected they would be able to do the addition using the decomposition to 10 strategy.

In the main activity, the egg box was used to promote students using the decomposition to 10 strategy since it allowed students to use the groups of 10 structure. We conjectured by exploring the groups of 10 in an egg box students would acquaint the number pairs that make 10, or the friends of 10. Next, students played the "finding the friends of 10" game in which they had to find their friends of 10. In this activity, the friends of 10 is done in a more abstract way without the egg box. We expected the egg box activity have given a strong base for students to work with the friends of 10. We analyzed students' video recording and written works in this lesson.

Activity 4.1: Review on math rack

Before students worked on the main activity, they repeated the previous lesson on the math rack. The aim of this activity was to review the previous activity with the math rack in which students have explored the structures in it and used them for showing a number. In this activity, students repeated the activity and we observed what structures they would use. We conjectured students might use double and groups of 5 structures.

The teacher opened the lesson by reviewing the using of math rack. She asked some students to show a number by using the math rack in front of the class. Here, we observed the strategies used by the students. The following fragment shows the interaction between the teacher and students during the activity.

Teacher	:	Who are ready for learning mathematics?
Students	:	I'm ready (raising their hand).
Teacher	:	Are you happy learning math?
Students	:	Yes, happy.
Teacher	:	Ok, before we start the lesson today, let's repeat what we did yesterday. Yesterday you did some counting by using what tool?
Students	:	Math rack.
Teacher	:	Where it is? Do we have it in our class room now?
Students	:	There it is.
Teacher	:	Salma, could you hand me the math rack please?
Salma	:	(Step forward and give the math rack to the teacher).
Teacher	:	Ok, we're going to use this now. Andini, come here and show me 5.
Andini	:	(Moving the first ball, second ball, and 3 balls together).
Teacher	:	You were still counting one by one, weren't you?
Andini	:	(Nod her head).
Teacher	:	Ok, thank you Andini. Now, Fariz, come here. Show me 8.
Fariz	:	(Moving the balls one by one until he got 8).
Teacher	:	Is this correct?
Students	:	Yes.
Teacher	:	Yes this is 8, but it still took a long time. Do we still need to count one by one?

Students	:	No.
Teacher	:	Ok, Now Vika. Show me 4.
Vika	:	(Moving 4 balls at once).
Teacher	:	Show me another way of 4.
Vika	:	(moving 2 balls in the upper bar, and 2 balls in the bottom bar).
Teacher	:	Yes, very good. Ghea, come here and show me 6.
Ghea	:	(Moving 3 balls in the upper bar, and 3 balls in the bottom bar).
Teacher	:	You see what Ghea just did? She used the double strategy and it's fast. 6 is also in the song. Tell me, what plus what?
Students	:	3 plus 3.

The teacher started the lesson by conditioning the students to be ready for the lesson. This is an evidence of the socio norm in this classroom where students were trained to get themselves prepared for the lesson. As usual, the teacher repeat the previous lesson, she wanted to promote the students to use the structures in a math rack.

Our observation showed that some students still used counting all when using the math rack. When asked to show 5, Andini did not necessarily use the groups of 5 in the math rack. She still moved the first and the second ball one by one, and after that, she probably realized that she needed 3 more balls, so she moved 3 balls at once. Andini was not aware of the structures and thus she used counting all. Fariz also showed the same unawareness, when asked to show 8, he moved the balls one by one until he got 8. He did not use the structure at all. This indicated that not all students have understood the structures in a math rack. The activities in the previous lesson might have not been fully meaningful for Andini and Fariz.

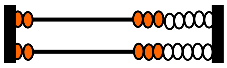
The teacher gave a comment on Fariz's strategy that it still took a long time to show an 8, but she did not show other strategies of showing 8. This could have been a learning moment for Andini and Fariz if they had had the chance to learn other strategies that were more effective. For our next HLT, this observation gave an input for the teacher for improving the socio norm by giving an immediate feedback to the students.

Vika and Ghea used double structures to show 4 and 6. This indicated that they have understood about some double sums since they were able to decompose 4 into 2 and 2, and 6 into 3 and 3. Vika and Ghea might have been influenced by the double song.

The following fragment showed students' strategies of using a math rack on addition problems.

Teacher	:	We can use this math rack to help us doing addition. Wira, come here. Can you show me 4?

Wira : (Moving 2 balls in the upper bar and 2 balls in the bottom bar)
Teacher : Let’s add 4 more. 4 plus 4.
Wira : (Moving 2 more balls in the upper bar and 2 more balls in the bottom bar).



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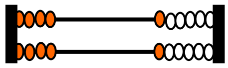
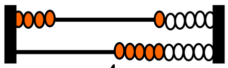


4 + 4

Teacher : So what is it?
Wira : 8.
Teacher : Is there any other way?
Dinda : Yes .
Teacher : Ok Dinda, come forward and show us your way of doing 4 + 4.
Dinda : (Moving 4 balls at once and then another 4 balls in the upper bar/5 orange and 3 white).



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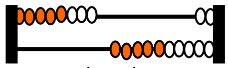
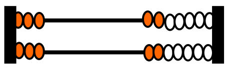


4 + 4

Teacher : Ok, but you still count one by one, right?
Dinda : (Nod her head).
Teacher : I have another question. Ais, come forward please.
6 + 6.
Ais : (Moving 3 orange balls in the upper bar, and 3 orange balls in the bottom bar. Then adding 3 more balls in the upper bar and 3 more balls in the bottom bar.



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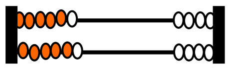
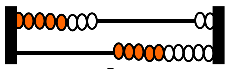


6 + 6

Teacher : So what is it?
Students : 12.
Teacher : How did you figure it out so quickly?
Students : (Unclear voice).
Teacher : You see, there are 10 oranges and 2 whites, so altogether is 12.
One more time. Raihan.
8 + 8.
Raihan : (Moving 8 balls in the upper bar at once and 8 balls in the bottom bar at once)



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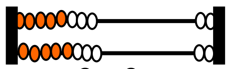


8 + 8

Teacher : What is it Raihan?
Raihan : (inaudible).
Teacher : Look at the orange balls, there are 10, six white balls. Altogether is 16.

In this fragment, the teacher only gave double sum problems. She started with 4 + 4, Wira used double structure to solve it. Not only did Wira use the double structure to solve the addition problem, but also used it to show 4. He decomposed it into 2 and 2. This indicated that he has known some double sums. Unlike Wira, Dinda only used the balls in the upper bar. However it was not clear whether she used the structures or not, since the teacher did not ask her how she did the addition. The teacher made a direct judgment that Dinda were still using counting all and did not discuss the difference of the two strategies. She could have asked Dinda how she did the counting, and helped her to understand the structures by asking questions such as how many there are? How many orange balls and how many white balls? This could have been a learning moment which allowed students to see the groups of 5 in Dinda's strategies (i.e., 4 + 4 = 5 + 3 = 8).

In the 6 + 6 problem, the double structure became more visible. Ais used double structure as he put 6 balls in the upper bar and 6 balls in the bottom bar. However, he could not use the structures in the math rack to solve the addition problem. The teacher immediately asked the students to tell the answer, but no body gave an explanation, then she showed that there are 10 orange balls and 2 white balls so altogether is 12. The same occurred when students worked on 8 + 8. Raihan could show 8 directly, but not necessarily able to do 8 + 8. He could not use the structures in the math rack. The teacher explained that there are 10 orange balls and 6 white balls, so altogether is 16. For this two examples (6 + 6 and 8 + 8), the teacher could have asked students how many orange balls and white balls separately and how many all balls together so that students were stimulates to see the group of 10 structure.

Overall, the observations showed that students have not employed the math rack to show their thinking process while working on addition problem. Students did not fully understand the structure in a math rack (i.e., the groups of 5, groups of 10 and double structure) and did not know how to use them in doing addition up to 20. Up to this moment, students have been able to show a number by using a math rack but not to do addition. We drew some conclusions that the introductory activity of exploring the structures in a math rack did not provide a strong base for students in a sense that they haven't fully conceive the idea of using the structures in a math rack to solve addition problems up to 20.

Activity 4. 2: Friends of 10 in the egg box

After a reviewing activity of using a math rack in doing double sums, the teacher started a new topic which is the friends of ten. For this activity, we proposed using the egg box in out HLT since the egg box gives possibilities for students to use groups of 10 structure. We used a worksheet in which students would write down the number of eggs and the number of missing eggs which altogether would make 10. We expected that students would generate an understanding of number pairs that make 10 or the friends of 10.

The observations showed that most of the students could immediately tell the number of the eggs and the number of the missing eggs. They also could write down the addition between the eggs and the missing eggs. For example, when there were only 2 eggs in the box, students could tell that 8 eggs are missing, and they could write 2 + 8 = 10. The teacher then stuck the egg box cards on the white board and asked some students to write the addition of the eggs and the missing eggs and then she used cards to discuss the friends of 10.

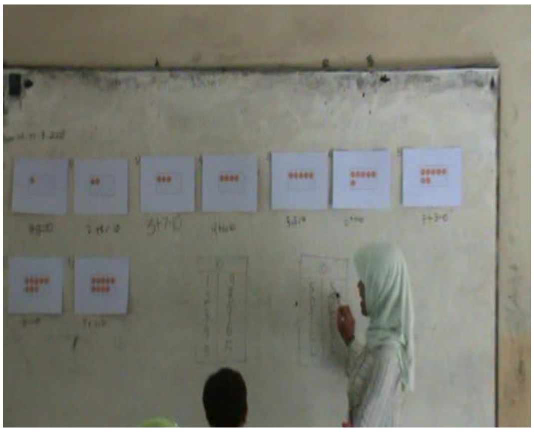


Figure 5.11: Friends of 10 in an egg box

After that, she reminded the students of the trick they learned in grade one (i.e., the first letter trick). In Indonesian language, each pair of numbers has the same initial letter which allows students to memorize the friends of 10.

10	
1	9
2	8
3	7
4	6
5	5

10	
Satu (One) *S*	*Sembilan* (Nine) *S*
Dua (Two) *D*	*Delapan* (Eight) *D*
Tiga (Three) *T*	*Tujuh* (Seven) *T*
Empat (Four) *E*	*Enam* (Six) *E*
Lima (Five) *L*	*Lima* (Five) *L*

Table 5.1: the first letter trick

We observed that students have become familiar with the groups of 10 structure in an egg box as they could tell immediately the number of eggs and the number of missing eggs.

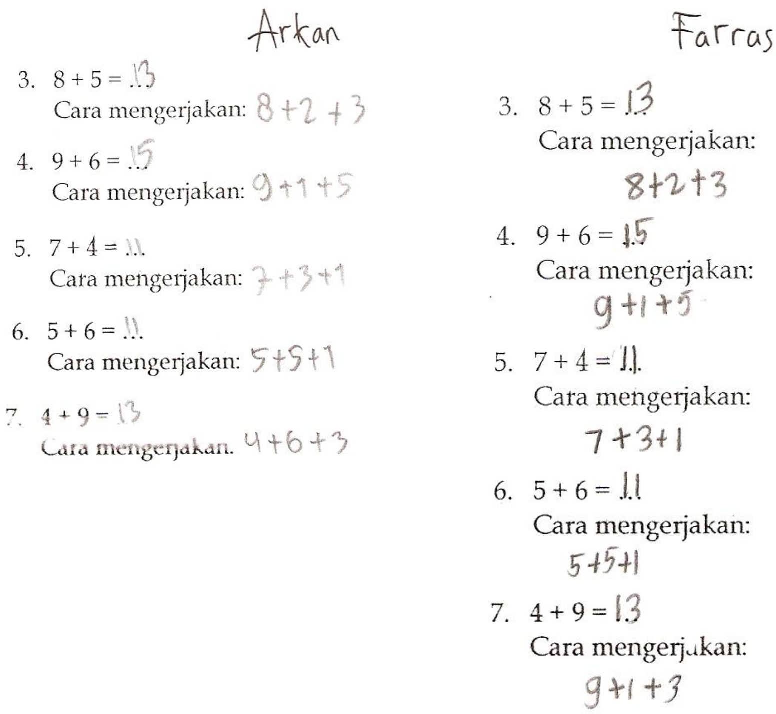
Activity 4. 3: "Find the friends of 10" game

In this activity we wanted to see students' understanding of friends of 10 without using the egg structure anymore. We hoped students would be able to use the friends of 10 in a more abstract way, that is in the "find the friends of 10" game. In this game, students got a random number from 1 to 9 and they had to find a partner who held the other number that makes 10. We expected that students would have no difficulties playing this game since they already worked on the egg box and the first letter trick.

The observation showed that students were happy to have a physical activity, since they were moving around searching for a friend. However, some students did not participate actively, since they would just stand still and someone else would come to them. We suspected that these students did not really understand the rule of the game, they looked confused when their friends were running and calling for a friend. One student had got a different idea of the game. He was holding a number 1, and instead of looking for a friend who held 9, he was looking for a zero. He thought, that he had to make a 10, thus a one and a zero. These observations showed how an actor point of view contributes in the activity. It implies that in out next HLT, we need to make a clear instruction for the students.

Our expectation that all students would immediately know the friends of 10 was not fully achieved in this activity. The large number of students and the capacity of the room might have caused such failure. This game would have been better played in a smaller group so that students have fewer choices of friends.

The game was directly followed by an activity in which students worked on a worksheet. In the worksheet, we gave some addition problems to see whether the egg box activity and the "find the friends of 10"game gave a significant effect on students' understanding of decomposition to 10 strategy. We chose some students' work and looked at a part of the worksheet that gave us information about students' strategies.


Arkan

3. 8 + 5 = 13
Cara mengerjakan: 8 + 2 + 3

4. 9 + 6 = 15
Cara mengerjakan: 9 + 1 + 5

5. 7 + 4 = 11
Cara mengerjakan: 7 + 3 + 1

6. 5 + 6 = 11
Cara mengerjakan: 5 + 5 + 1

7. 4 + 9 = 13
Cara mengerjakan: 4 + 6 + 3

Farras

3. 8 + 5 = 13
Cara mengerjakan:
8 + 2 + 3

4. 9 + 6 = 15
Cara mengerjakan:
9 + 1 + 5

5. 7 + 4 = 11
Cara mengerjakan:
7 + 3 + 1

6. 5 + 6 = 11
Cara mengerjakan:
5 + 5 + 1

7. 4 + 9 = 13
Cara mengerjakan:
9 + 1 + 3

Figure 5.12: Students' written work

Students' written work showed that there were only 6 of 37 students that used the decomposition to 10 strategy while the rest of the class wrote formal notation of solving addition problems. A formal notation is the written procedure students learned to solve addition problems. Usually this procedure is used to solve 2 digits addition problems when first students adding the ones and then the tens. For many students, additions should be done by using this procedure, not by using their own way of thinking. We observed that students use this procedure which clearly does not make any sense for addition up to 20 since the written procedure does not explain students' way of thinking.

Vertical standard written procedure | Horizontal standard written procedure

Figure 5. 12: Vertical and horizontal standard written procedure

The using of formal notation does not mean that students have reached a formal mathematical level, since many students were actually counting by fingers.

Arkan and Farras used decomposition to 10 strategy, they have shown an understanding of splitting an addend to make 10 and add the remainder (Figure 5.12).

As we have described about the steps of doing the decomposition to 10 strategy, this lesson has covered only the first and the second steps. Our observations showed that the structure in an egg box has served very well in introducing the friends of 10. However we found a big jump from the friends of 10 to the decomposition to 10 strategy as students did not immediately used the friends of 10 to do the decomposition to 10 strategy.

We concluded that the "find the friends of 10" game did not support the using of the decomposition to 10 strategy in solving addition problem. We identified, that students did not immediately move from knowing friends of 10 to using it for solving addition problems. The reason is probably that there is another step they should do when using the decomposition to 10 strategy, which is the splitting of the other addend, for example, 8 + 5 = 8 + 2 + 3 = 13. In order to help students perform this splitting, in the next lesson we will use the math rack. We conjectured, the splitting can be seen clearly by the movement of the beads.

5. 5. Lesson 5: Friends of 10 strategy (20 August 2008)

In the previous activity students have done some activities on decomposition to 10 strategy but we concluded that they haven't fully grasped the concept of the 4 steps and haven't been able to use it in doing addition up to 20. Even though students have known the number pairs that make 10, or the friends of 10, they still haven't been able to use it in the decomposition to10 strategy since it required one more step in which students have to split the other addend. This activity aimed at reinforcing students' conception about the decomposition to 10 strategy, more precisely on the splitting of the other addend by using the math rack.

The teacher would start the activity by reminding the students to "the sum I know" worksheet and focused on the 10+ sums (e.g., 10 + 1, 10 + 2, etc) and or the +10 sums (e.g., 1 + 10, 2 + 10, etc). Realizing that doing 10+ and +10 sums are easy, students would be guided to use those sums to solve addition up to 20 problems. Next, students learned to decompose one addend so that they can perform the decomposition to 10 strategy. For example: 8 + 6, student would learn to decompose 6 into 2 and 4, because 2 is the friend of 8, then they would be able to perform 8 + 6 = 8

+ 2 + 4 = 10 + 4 = 14. This decomposition would be seen clearly by using a math rack. We hoped students would understand the decomposition to 10 through moving the beads in a math rack.

Activity 5.1: Reviewing the "The sum I know" worksheet

The teacher started the lesson by reviewing the "The sum I know" worksheet, more precisely on the 10+ and +10 sums. The teacher asked the students who have known those sums, and the observation showed that many students knew the 10+ and +10 sums. This was indicated by the work they did in "the sum I know" worksheet. The teacher used this fact to guide the students in utilizing these sums when working with addition up to 20. First, she reviewed the first letter trick as a support for finding the friends of 10, and then asked students the 10+ sums. Students answered the 10+ sums immediately and they also said that those additions were easy.

Knowing the 10+ sums, students were guided to do the decomposition to 10 strategy. The following fragment shows the interaction happened in the classroom.

Teacher	:	Who still don't know 10+ additions?
Students	:	(no one raising their hands).
Teacher	:	So, the additions of 10 are what? What do you think?
Students	:	Easy.
Teacher	:	Ya, that's why we use this (pointing the first letter table).
Students	:	SS, DD, TT, EE, LL.
Teacher	:	Number pairs that make 10 when added. Now I will ask, 10 + 7 is what? (writing 10 + 7 on the board).
Students	:	17.
Teacher	:	Ok. Now, 9 + 8 is what?
Students	:	17.
Teacher	:	How did you know it so quickly?
Gina	:	Because 9 + 9 is 18. We take away 1, so it's 17.
Teacher	:	You used the table. Is there any other way? Wira?
Wira	:	9 needs 1 more to get to 10. We took 1 from 8, so 7 is left. 10 + 7 is 17.
Teacher	:	Write your answer on the board, please?
Wira	:	(writing 9 + 1 + 7 = 17).
Teacher	:	Why is it 9 + 1 + 7?
Students	:	(everybody has an answer).
Teacher	:	Let's listen to Bintang.
Bintang	:	Because 9 + 1 is 10. 10 + 7 is 17.
Teacher	:	Yes 9 + 1 is 10 and we add 7 so it's 17. Your parents might tell you that when you do the addition like this, 9 + 8. You keep the 9 hold, and you use your fingers to add 8. (showing 8 fingers) 9, 10. How many more fingers are left?
Students	:	7.
Teacher	:	So, 10 + 7 is?
Students	:	17.
Teacher	:	Yes. You don't need to count one by one, 9, 10, 11, 12, 13, etc. It's too long. Now, I have 10 + 5. What is it?

Students	:	15.
Teacher	:	Yes, good. How about 7 + 8. Fathur?
Fathur	:	(writing 15).
Teacher	:	How did you get that. Write your way.
Fathur	:	(pause).
Teacher	:	Use this (pointing the first letter table).
Fathur	:	(writing 7 + 3 + 5 = 15).
Teacher	:	Is this correct?.
Students	:	Yes.
Teacher?	:	How come? What's the pair of 7?
Students	:	3.
Teacher	:	So 7 + 3 + 5 = 15. Do you understand?
Students	:	Yes.

In this fragment, the teacher asked the students to find the friends of 10 referring to the first letter trick. This trick might help students decompose an addend. By knowing the friends of 10 of the initial number, students could do the next step of the decomposition to 10 strategy which is decomposing the other addend by subtracting the friends of ten from it. However, students did not immediately get this idea, Gina still used double strategy when solving 9 + 8. It was not a surprise since the question was an almost double problem and students have learned about double strategy. This indicated that double strategy has become more favorable for students.

Wira used decomposition to 10 strategy, his oral explanation indicated that he has understood the strategy very well. The observation we've done during the period of May and June also suggested the same indication. First, Wira found the friends of 10 of the first addend after that, he decomposed the other addend to find its remainder and added it to 10 for the final result. Wira showed that he knew all the steps mentally. Bintang explanation on 9 + 1 + 7 did not indicate his understanding of the strategy. We assumed that he had misinterpreted the strategy, as he knew that 9 + 8 is 17, and for him 9 + 1 + 7 was just another way of getting 17. The teacher did not revise Bintang's explanation while probably many other students still thought the same as Bintang. If the teacher had asked more questions to get students' thinking process, this could have been an important learning moment where the teacher could detect whether students have really understood the lesson or not.

The decomposition to 10 strategy could also be done by using fingers. Since many students used counting on with the fingers to solve addition problems, the teacher showed a faster way counting on with fingers. Student normally kept the first addend hold, and showed the other addend by their fingers and then counting on. Instead of counting the other add end one by one, students could stop counting when they have reached 10, and after that, they could see how many fingers were left and add them to the 10 and got the final result. For example 9 + 7, students kept the 9 hold, and showed 7 fingers. The counting on were done only until 10, so one finger was left out. Students still got 6 remaining fingers, and 10 + 6 is 16. With this way, teacher suggested a faster way of counting with fingers rather than counting on one by one. However, a deeper attention was not given to this strategy of counting. In the observation, we did not see students used this nor the teacher promoted them to do so. This strategy was in line with the decomposition to 10 strategy, therefore we thought it would be better if students learned it when they were still in the level of counting on with fingers. This strategy might help students' transition from counting on with fingers to a more formal strategy without counting.

Fathur knew 7 + 8 was 15 but he had difficulties of writing his way of thinking. It was possible that he had thought a strategy and that he thought it was different from what the teacher expected, and thus he was reluctant to write his own strategy. Or he might have used the decomposition to 10 in his head but had difficulties of writing it down. The teacher gave him a help by telling him to use the first letter trick. This trick has helped him find the friend of 7, which is 3. After that he wrote 7 + 3 + 5 = 15.

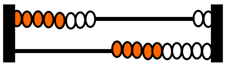
In this activity, we observed that the first letter trick has allowed students to find the friends of 10 easily. However, the decomposition of the other addend remained unclear. For example, when discussing 9 + 8 students use 9 + 1 + 7 why not 9 + 5 + 3? The essential concept of making the addition into 10 and then splitting the other addend was not clearly visible in this activity. Having more discussion might help make the concept more visible, thus we will consider to put it in our next HLT.

Activity 5. 2: Decomposition to 10 strategy by using a math rack

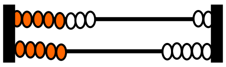
The next activity was using a math rack in doing the decomposition to 10 strategy. The teacher held a giant math rack in front of the class and asked some students to perform the addition by using the math rack. We observed students' learning in the following video fragment.

Teacher	:	We can also do additions by using a math rack. Ok, who can perform addition using a math rack? Fikri, come forward please. Find 8 + 4 by using this math rack.

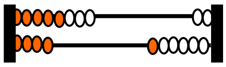
Fikri : (Thinking).
(Moving 8 balls at once).



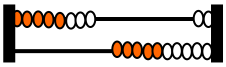
(Thinking).
(Moving 5 balls at once).



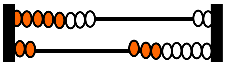
(removing 1 ball).



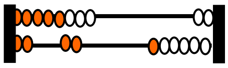
Teacher : So, what is it Fikri?
Fikri : (pause).
Teacher : Why? Why can't you see it easily?
Dira, do you have other way? Come here and show it.
Dira : (Moving 8 balls at once).



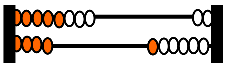
(Thinking).
(Moving 2 balls at once).



(moving 2 more balls).

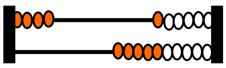


Teacher : (Putting 4 balls in the bottom bar together).

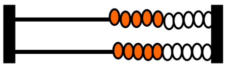


Is it the same as Fikri's?

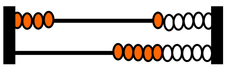
Students : Yes.
Teacher : We can't see easily how many balls because 8 are in the first bar and 4 in the second bar. Who has different way? Andina.
Andina : (moving 4 balls).



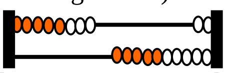
Teacher : (Removing the balls back).
8, not 4.



Andina : (moving 4 balls).



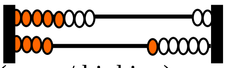
Teacher : Why 4?
Andina : (Moving 8 balls).



(hesitating, want to change the arrangement of the balls).

Teacher : Yes, 8.
Now plus 4.

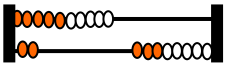
Andina : (Moving 4 balls).



(pause/thinking).

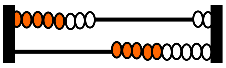
Teacher : Remember the pairs? SS, DD, TT, EE and LL. What do we use that for? Come here Adin.

Adin : (moving 10 balls in the upper bar, and 2 balls in the bottom bar).

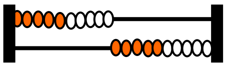


Teacher : Why do you directly go to 10?
First, show the 8.

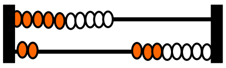
Adin : (moving 8 balls).



(moving 2 more balls).



(moving 2 balls in the bottom bar).



Teacher : Is this correct?

Students : Yes.

Teacher : What is the result?

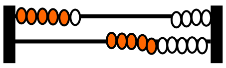
Students : 12.

Teacher : How can you know that so quickly?

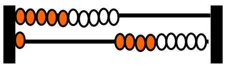
Students : Because there are 10 and 2.

Teacher : Now you understand?
First Adin had 8, and then she completed it into 10. Since she has already used 2, she added 2 more, and had 10 + 2 = 12.
Ok, now Tyo. Come here.
6 + 5.

Tyo : (moving 6 balls).



(moving 4 balls and 1 ball).



Teacher : What is the result Tyo?

Tyo : 11.

Teacher : Is that correct?

Students : Yes .

In this video fragment, we found some interesting observations. At first, students did not know how to use the math rack properly. They used the math rack as a tool to represent the numbers being added but they did not use the rack to support their thinking. When showing 8 + 4, Fikri just took 8 balls from the upper bar and 4 balls from the bottom bar. The balls on the math rack did not help him determining the result of 8 + 4 easily since the balls were not structured in groups of 5, groups of 10 or double. Even though Dira seemed to have a different idea, but she used the same way as Fikri.

Andina was probably had a different way, when she took 4 balls, we assumed that she would use double. However, the teacher did not let her finished her work; instead she gave an immediate feedback by repeating the question. The teacher sent a message that Andina should show an *eight* instead of a *four*. At the end, Andina used the same arrangement as Fikri and Dira. The teacher's intervention disadvantaged students' learning because Andina could have shown a good strategy and other students might have learned from Andina. In this fragment, students used the math rack only to represent the numbers that are added. The math rack did not serve as a tool to support their thinking process. Having this misunderstood, students did not employ the structure in the math rack to help them doing addition up to 20.

Adin performed very well when she showed how she added 8 + 4 by first making the 8 into 10 and then adding 2 more. By doing this way, the groups of 10 structure could be seen clearly, and students could reason that there were 12 because 10 balls are in the upper bar and 2 more balls are in the bottom bar. Adin has stimulated the other students to do the addition in this way. Tyo did not find any difficulties solving 6 + 5. He added 4 to the 6 to get to 10 and added 1 from the remainder so he had 11 as his final answer. We observed that students did not have any difficulties determining the 10+ sums which were the effect of doing the friend of 10 strategy, however they still haven't fully grasped the idea of decomposing the other addend. The reason for this might be that students have never had an opportunity to explore the structures in the math rack for doing the decomposition to 10 strategy. The activity with the math rack was designed to enable students to do so, but it did not succeed which might have been caused by the absence of students' discussion. If Adin had explained her reason of doing the addition, the other students might have learned the idea of decomposition to 10 instead of just copying Adin's strategy.

Next, all students got an opportunity to use the math rack themselves. Every two students received one math rack, and they had to use it after one another. We observed two students; Bintang and Faraz. In the observation we looked at how they worked with the math rack, and what structures they used during the lesson.

First question was 6 + 8. Bintang had the first turn. He was still confused when representing the numbers with the beads, thus he used counting all. He counted 6 beads, and added 8 more. He got 10 beads in the first bar, and 4 in the second bar. However, he did not use the groups of 10 structure. He still counted all. This indicated that he still hasn't understood the structures in a math rack. For him, the math rack was only an instrument to show the numbers being added.

The second question was 7 + 7, this time Faraz got his turn. He moved 7 beads at once and then moved the other 3 beads in the first bar. Then he took 7 more beads in the second bar, but immediately changed it by taking only 4 beads. He immediately raised his hand and did not show any indication that he was doing counting all. Looking at his performance, we assumed that he has been able to use the groups of 10 structure in the math rack.

Bintang got another turn at the third question, and it was 8 + 6. This time, he was struggling to do it faster than the first time. Bintang tried to use the groups of 5 structure. First, he moved 5 beads and then he moved another 5 beads, giving him 10 beads in the first bar. He was thinking hard how many more he should take, we assumed that he did counting all or counting on by heart. After a while, he took 4 beads in the second bar. Clearly, Bintang did not use the math rack to help him solve the problem; instead he used it only to show the answer to the problem.

Bintang was still struggling how to use the math rack, he reshuffled the beads and started all over again. Gina came and helped him by showing how the addition should be done. First she took 8 beads, and then did the counting on by moving the beads one by one and saying the 1, 2, 3, 4, 5, 6. By doing this, they got 10 beads in the first bar and 4 beads in the second bar and 14 as the final result. Gina's explanation has helped Bintang improved his understanding. He showed a gesture of an understanding, however we did not find any indication if he used the group of 10 in his strategy.

The next question was done by Faraz, it was 9 + 5. This was a good question because it promoted students to do the decomposition to 10 strategy since 9 was close to 10. Faraz took 9 beads and then took 1 more which gave him all beads in the first bar. He moved to the second bar and took 4 beads. After that he immediately raised his hand, showing that he has got the answer. Now we could be sure that he was using the groups of 10.

Next, the teacher wrote on the white board how their activity could be written in a formal mathematics. The guidance given by the teacher is shown in the following fragment

Teacher	:	What did you do first? After getting 9 beads, how many did you add to it?
Students	:	1.
Teacher	:	(Writing 9 + 1). Then how many more did you add?
Students	:	4.
Teacher	:	(Writing 9 + 1+ 4). Why? Because these together are…
Students	:	10.
Teacher	:	Yes, there are the SS.

In this fragment, the teacher started the transition from using a math rack as a tool to represent the numbers that were being added and the strategies used when adding to a formal mathematics. She wrote the mathematical expression corresponded to each step done by the students, and produced a formal written mathematical way of solving the problems. This was a critical moment in students' learning process because this has initiated students to move from concrete object to a formal mathematics. However, we thought it would have been better if the explanation was done by the students themselves.

The next question was written in the white board, it was 7 + 9. Bintang was still struggling when solving this problem. The math rack did not help him solve the addition. He counted the beads one by one before took 7 beads, and then he move the other 3 beads left in the first bar. After that, he looked puzzled trying to determine how many more should be added in the second bar. This indicated that he was still doing counting on in his head, and he used the math rack just to show his final result. Gina helped him again by showing how the addition should be done. After getting seven beads, she moved the other beads while counting one by one until she reached 9. However, Gina did all the work, not Bintang. Before giving the last question, the teacher repeated the explanation of the written formal mathematics.

Teacher	:	Before we continue to the next question, everybody please look at the board. This one 9 + 5.
Gina	:	9 + 1, which is S and S.
Teacher	:	Yes, 9 and 1, S and S. What remained from 5?
Students	:	(busy talking) Gina said 4.
Teacher	:	9 + 1 is …

Students	:	(busy talking) Gina said 10.
Teacher	:	And 10 + 4 is…
Students	:	(busy talking) Gina said 14.
Teacher	:	Now, 7 + 9. Look, 7 is T so what is its pair?
Students	:	(busy talking) Gina said 3.
Teacher	:	7 and 3. And what's left from the 9?
Students	:	(busy talking) Gina said 6.
Teacher	:	So what do you have now?
Students	:	(busy talking) Gina said 10 + 6.
Teacher	:	Yes, what is it?
Students	:	(busy talking) Gina said 16.
Teacher	:	Have you all understood?
Students	:	(busy talking) Gina said yes.
Teacher	:	Ok, now I'll give you the last problem.

In this fragment, the teacher explained the procedure of doing the decomposition to 10 strategy. First, find the friends of 10 and then decompose the other addend and finally do the decomposition to 10 strategy. However, we observed that students were not engaged during this explanation. Most of the students were busy talking and playing with the math rack. Gina was the only one who showed an interest in the explanation. The teacher was not aware to this classroom situation as she did not give an immediate warning to the students to get their attention on the topic she explained. This observation showed us another classroom socio norm that when students did not know what to do with the math rack, they tend to lose their attention on the lesson.

The last question 8 + 5, was done by Faraz. He took 8 beads and then 2 more beads in the first bar. Next he took 3 beads in the second bar.

The teacher asked the students how to write the addition.

Teacher	:	What is the result?
Gina	:	13
Teacher	:	How do we write it? 8, what is its pair?
Students	:	D-D, 2
Teacher	:	8 and 3, and plus how many more?
Students	:	3
Teacher	:	And 10 + 3.
Students	:	10 + 3 = 13
Teacher	:	Ok good.

Based on our observation in this lesson, we concluded that the using of a math rack as a tool for visualizing students' thinking was a complicated process for students. They did not immediately discover the way of using the math rack instead, what happened was the other way around. Some students solved the additions by counting on in their head, and then used the math rack to show the result of the additions. In this case, students were only using the math rack as another way of representing a quantity, not representing way of thinking.

In our next HLT we will propose that the teacher asks one students to tell each step s/he does when doing friends of 10 with the math rack and write it in the black board. Then students might have a discussion of translating their action into a written formal mathematics expression.

5. 6. Lesson 6: Addition up to 20 using a math rack (21 August 2008)

This was the last activity in our experimental phase. The goal of this lesson was to help students understand the double strategy and decomposition to 10 strategy, and also the flexibility of using these in solving problems of addition up to 20. The teacher would start the lesson by discussing the difference between the two strategies. Students would be given some problems to be solved by using both double and decomposition to 10 strategies. We expected, that they would compare each strategy for each problem given and finally would be able to choose the best strategy for a certain problem. For example, almost double sums are best to be solved by the double strategy, while sums such as 9 + 6, 8 + 5 will be better solved by decomposition to 10 strategy since one of the addend is close to 10. In order to do this, the teacher used a big math rack which is made from circled papers attached together by a thread. The paper-thread math racks were hung on the white board so that all students could see them. Next, students would use the math rack in solving addition up to 20 problems. We hoped, as a result of this lesson, students would finally understand the using of a math rack as a tool to represent their thinking process. At the end of the lesson students did a test as an assessment of what they have learned during the experiment.

Activity 6. 1: Reviewing double and decomposition to 10 strategies on a math rack

The teacher started the lesson by settling down the class, preparing the students to get ready for the lesson. When students were ready, she reviewed the double strategy and decomposition to 10 strategy. The following fragment shows the interaction in the class.

Teacher : Look, what is this?
(moving 6 beads, 5 blue and 1 white)
Students : 6.
Teacher : How can you know it so quickly?

Gina	:	Because at the upper bar, there are5 blue, ….
Teacher	:	Do we need to count one by one?
Students	:	No
Gina	:	And 1 white
Teacher	:	Yes, no need to count 1, 2, …, 6. We can see it right away 5 and 1 is 6. Now, 6 plus ….. (moving 7 beads in the bottom bar) what is this?
Students	:	7.
Teacher	:	So what is the result?
Students	:	13.
Teacher	:	Ok, Kasya could you tell how come you get 13?
Kasya	:	Because 6 + 7 = 13.
Teacher	:	Yes, but why you can find it out so easily just by looking at this math rack?
Kasya	:	Because there are 5 blue, and also 5 more blue below. 5 + 5 is 10. 10 + 3 = 13.
Teacher	:	Yes. These are 10. 5 blue at the top, and 5 more at the bottom. There are 3 white, 1 at the top, and 2 at the bottom. So, 6 + 7 = 13. This way is the double. Can you figure out other way of double?
Naga	:	(about to move the beads at the other side of the upper bar to make a 10)
Teacher	:	No, those are staying there. We only use these beads. Do you see any other double? Kasya has used 5 and 5 and 3 more. But is there any other double?
Naga	:	(puzzling)
Teacher	:	Find the ones who have pair. Can you do that?
Naga	:	(puzzling)
Teacher	:	Double could also mean the same. Wira?
Wira	:	6 + 6 = 12, and there is 1 more bead so 12 + 1 is 13.
Teacher	:	Yes, Wira said to double these. Look. These are the same, 6 on top and 6 on the bottom. That's the double. What is 6 + 6?
Students	:	12.
Teacher	:	12 + 1?
Students	:	13.
Teacher	:	So, we have 2 kinds of double. It can be 5 + 5, 10. And we add 3 more is 13. or it can also be 6 + 6, 12 and add 1 more is 13.

The teacher asked students to tell the number represented in a math rack. First she moved 6 beads in the upper bar of the rack, and students immediately could tell that it was 6. Gina said that it was because there were 5 blue beads and 1 white bead. This showed that she used the groups of 5 structure. The teacher encouraged the students to always use the structures in a math rack and not to count the beads one by one. This indicated that she realized that some students in her class were still counting one by one, thus, by showing the structures, she helped them to understand the structures.

At first, Kasya looked like she did not really understand the question given by the teacher, as she said 6 plus 7 is 13 because 6 + 7 = 13. Here, she might did not see the math rack as a tool to support her thinking process since she did not link her answer to the structures in the rack. Then the teacher gave her another leading question which promoted her to look at the structures of the rack. She could use groups of 5 structure, as she saw 5 blue beads at the top bar, and 5 more blue beads at the bottom bar, so 10 blue altogether. Ten blue beads and 3 white beads together were

13. Kasya’s answer showed that she has used the groups of 5 structure as it was clearly visible at the math rack.

When the teacher asked the students to show another way to do the double strategy, Naga looked confused. This indicated that Naga still has not really understood about doing double strategy in a math rack. The structures in the rack did not directly lead him to use the double structure. We assumed that there are other students beside Naga who could not see the double in a math rack which might have been caused by the lack of understanding of the flexibility of the structures in a math rack.

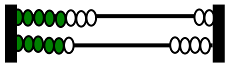
The teacher gave him a hit by asking him to find a number that was contained in both the upper bar and the bottom bar. This hint did not help Naga as he was still puzzling. This indicated that he still did not understand the idea of the double structure. We also observed that the teacher did not distinguish the double structure and the groups of 5 structure. When students worked with groups of 5 strategy, they took away 5 from both addends and composed a 10, thus, the teacher considered it as the double strategy because students doubled the 5. In our next HLT, we need to clarify this distinction so that students would understand the idea of double strategy.

Wira found other doubles in the rack. First, he used 6 + 6 and got 12, and then he added 1 more bead that are left which gave him 13 as the final answer. The teacher clarified Wira’s strategy by demonstrating the structures in the math rack. She moved 6 beads in the upper bar, 6 more at the bottom bar, and explained that those beads make a double structure.

Next, the teacher gave another problem 8 + 6, she moved 8 beads in the upper bar, 6 beads in the bottom of a math rack. The following fragment showed the learning of the students.

Teacher	:	(Moving 8 beads in the upper bar of a math rack) What is it?
Students	:	8.
Teacher	:	(Moving 6 beads in the bottom bar of a math rack) What is it?
Students	:	6.
Teacher	:	Can you see the result clearly in the rack?
Students	:	14.
Teacher	:	Fikri, how could you know that? How did you use the rack?
Fikri	:	8 + 6 is 14 because 9 + …. (pause) 5 is 14.
Teacher	:	Ok, I’ll use the big rack here on the white board so that all of you can see it. Ok, this is the double strategy. 8 + 6. Tyo, come here and show 8 + 6.
Tyo	:	(Trying to move 8 circles at once).
Teacher	:	Is it difficult to move them all together? Do it one by one then. Here, let me help you.

(The teacher and Tyo were moving the circles one by one, until 8 circles)
Ok, Tyo has got 8 and 6. So what is 8 + 6?



Students	:	(Unclear voice) 16.
Teacher	:	16?
Students	:	14.
Teacher	:	How come? Vika, which doubles do you use?
Vika	:	(Moving forward in front of the class) (pause)
Teacher	:	Who can helps Vika to write down the way?
Students	:	5 + 5.
Teacher	:	Yes. 5 + 5. (writing “(5 + 5)”). So here is the double. (pointing at the green circles) and what?
Students	:	Plus 3 plus 1
Teacher	:	(Writing 3 + 1) equal to what? What is this? (pointing at “(5 + 5)”)
Students	:	10.
Teacher	:	10 + 4 is (writing “10 + 4 = 14”)
Students	:	14.
Teacher	:	So you write all the steps one by one. Now, who wants to do the other doubles? Here we doubled the 5.
Students	:	Me (raising their hands).
Teacher	:	Come here, Laras.
Laras	:	(Puzzled).
Teacher	:	What we just did was using these groups of 5 because they were the same. Now what do you want to use? Do you see any similarity?
Laras	:	(Pause) here we add 2 more. (pointing at the bottom bar)
Teacher	:	If you add 2 more there, what will you have? You separate the ones without any pair. Can you do that?
Laras	:	This one and this one.
Teacher	:	Ok, good. So what is now?
Laras	:	10 + 2.
Teacher	:	10 + 2?
Students	:	6 + 6.
Teacher	:	Ok. Bintang, can you help Laras?
Bintang	:	6 + 6.
Techer	:	Ok, Laras using this double. Is it the same as Vika?
Students	:	Noo ..
Teacher	:	Vika used the 5s . and now what does Laras use?
Students	:	6s.
Teacher	:	I see some of you are still confused. What does ‘double’ mean?
Students	:	The same
Teacher	:	Double means there are two equal groups. (pointing at the 6 circles). So what double does Laras use now?
Students	:	6 + 6 + 2 = 12 + 2 = 14. (the teacher writing “(6 + 6) + 2 = 12 + 2 = 14)
Teacher	:	Ok, can you do the double now?
Students	:	Yes.
Teacher	:	Ok. That was the double strategy.
A student	:	Miss, why do you use the bracket?
Teacher	:	Why did I use the bracket?
A student	:	Because it will be faster.
Teacher	:	The bracket is used to show the double. The identical numbers. Any other questions? So, double is when you have the upper bar is equal to the bottom bar. (pointing at the 14 circles) are these double?
Students	:	No. There are more on the upper bar.
Teacher	:	(moving 4 red circles away) Is it now double?

Students	:	Yes.
Teacher	:	Ok, Now you can see the difference. Can't you?
Students	:	Yes.

In this fragment, we observed students' struggle of using double structure to solve 8 + 6. When the teacher showed 8 beads and 6 beads on the math rack most of the students immediately tell how many beads were shown. We interpreted that students were able to tell a number less than 10 when shown by a math rack, moreover, we assumed that students use groups of 5 in doing so.

However, adding 2 numbers by using a math rack was not that easy for students. Some students could answer right away that 8 + 6 is 14, but when they were asked how they solved it in a math rack, they became puzzled. We assumed that these students have known 8 + 6 by heart and doing it by using a math rack was new for them. For example, Fikri did not use the structures in a math rack, instead he said 8 + 6 was 14 because 9 + 5 was also 14. This indicated that the rack hasn't been employed by the students to support their thinking. For them, the rack was used to show another way of getting the answer. This might have been because students did not work on the math rack themselves, instead they only looked at the math rack that was held by the teacher.

The teacher then moved to the white board and used the paper-thread math rack. This tool was used to replace a math rack, since it was handy in a sense that was easy to make, and big enough for the whole class. However, we observed that the paper-thread rack could not replace the math rack since the circles can not be moved all at once. When Tyo tried to move 8 circles, he could not do that, so that he had to move them one by one. The paper-thread rack could not be used to show students strategy when representing a number. We need to change this tool in our next HLT.

Vika still haven't understood the double strategy with the math rack as she did not know what to do with the rack. We assumed she did not understand the teacher's instruction. The word 'double' might not understandable for her. The teacher helped her by asking other students to help. One student suggested to double the 5, then the teacher wrote down the mathematical sentence that it is (5 + 5) + 3 + 1 = 14. We observed that some students could recognize the doubles of the 5 since the groups of 5 is clearly visible in the math rack.

Laras also looked confuse when she was asked to show another double. She did not understand the word 'double' and the teacher helped her by telling her to look for the same amount-beads in the two bars. Having this guidance, she wanted to add two more circles in the bottom line so that both line became equals. Laras's answer showed the actor's point of view, that double was when the two lines have the same amount of beads. The teacher showed which number was doubled by making a one on one correspondence of each of the circles in both lines. The double was shown by the circles that have pair at the other line.

For the decomposition to 10 strategy, the teacher asked Rara to show how to do 8 + 6 in front of the class. While Rara was arranging the paper, the teacher explained each step of decomposition to 10. The following video fragment illustrated the learning of the students.

Teacher	:	Ok, Rara. Come and show how to add 8 + 6 by using the number pair of 10.
Rara	:	(Moving 8 circles at the first line).
Teacher	:	Ok, now look what Rara just did. How many circles did she move?
Students	:	8.
Teacher	:	Ok, I'll write 8. Since we're working on the number pair of 10, so to get 10, how many more should we add?
Students	:	2.
Teacher	:	Ok, Rara, could you write it down?
Rara	:	(writing + 2 and then moving 2 circles)
Teacher	:	The question was 8 + 6, and Rara has added 2. So how many more? Let's count together
Students	:	(Rara was moving the circles) 3, 4, 5, 6.
Teacher	:	Yes, we have completed it. How many more did we just add?
Students	:	4.
Teacher	:	Let's add them together. 8 + 2 + 4 = 10 + 4 = 14. Do you understand?
Students	:	Yes.

In this fragment, we observed that doing decomposition to 10 was not as difficult as doing double. Here, we saw that the teacher showed each of the steps very clearly. Right after Rara collected 8 circles, she stopped her and show the students what Rara just did and what she should do next. After that students were guided to find the friend of 8, and they immediately knew that it was 2. This indicated that students have known the friends of ten since they had done some activity on that.

To show the composition of the other addend, the teacher asked the students to count the circles one by one. They have added 2 and to know how many more to add, students did counting on by saying 3, 4, 5, 6. After that, they saw that there were 4 circles in the bottom line. This strategy was very basic, since students were still using counting on. But this approach might help low achiever students who still did not know how to use the math rack.

Next, students worked on some addition up to 20 problems by using a math rack. Each of 2 students got a math rack, so that they had to work in turn. During this activity, we observed Bintang and Faraz. The first problem given by the teacher was 6 + 7 and she explicitly told the students to work with double strategy. We observed that Bintang was still using counting all as he moved each bead one by one. he got 6 beads in the upper bar and 7 beads in the bottom bar, but he did not show any indication of using the structures to find the result. The teacher asked the students what is the result, and they immediately answered that it was 13. When asked to explain where they got it, the students could argue that they got 13 from 5 + 5 + 1 + 2. Clearly, most students doubled the fives. This is an easy way of doing double in the rack, since the fives are clearly seen by its color.

The second problem was 7 + 9, the teacher still suggested the students to use double strategy. Faraz was looking at the beads for one second and then moved 7 beads at once. Then he put his finger at the 6^{th} bead and counted one by one until the 9^{th} beads before moving them. This indicated that he used groups of 5 to represent each number that were added. Next, he separated the white beads from the blue beads, giving him 10 white beads and 6 blue beads. When the teacher asked what was the result, students spontaneously said 16. Moreover, students explained that they got it by 5 + 5 + 2 + 4. After that, the teacher asked if there’s another way, and Dira said that she did 7 + 7 + 2. This observation signifies that when working with math rack, group of 5 was easier for students than double strategy.

Bintang got to do the next question, it was 8 + 4 and this time the teacher told the students to use decomposition to 10 strategy. First, Bintang moved the beads one by one until he got 8 beads, and then he moved the rest of the beads in the upper bar still by counting one by one. He was reciting the number each time he moved the beads. However, he did not do it properly since he counted until 4 but he moved 5 beads. This indicated that Bintang followed the way it had been done earlier when Rara and Adiza did the decomposition to 10 in front of the class.

The teacher asked the students to tell what the result was and how to get it. Students immediately answered 12 but not all students could reason by using decomposition to 10 strategy. A student said that it was 5 + 5 + 2, he might have still used the groups of 5 strategy. Naga argued it was because 8 + 2 + 2 = 10 + 2 = 12.

The next problem, 7 + 7 was done by Faraz. He moved 7 beads at once, and then moved the other 3. Then he went to the bottom bar and move 2 beads and 2 more beads. After that he raised his hand, indicating that he has found the answer. The teacher asked the students how many beads were in the upper bar, and the students answered 10, spontaneously. When asked how they got 10, students said it was 7 plus 3. After that, the teacher repeated the problem and wrote on the white board $7 + 7 = 7 + 3 + 4$.

So far, some students were able to use the math rack to show their strategy either the double or the decomposition to 10. Unfortunately, our goal to help students understand the flexibility of using the double or the decomposition to 10 was not achieved in this activity. The reason of this was because the class did not have enough time to practice on using the better strategy for certain problems. For example, 7 + 8 would probably be better solve by using double, while 9 + 5 could easily solved with decomposition to 10. Having the ability of choosing which strategy to use was not achieved in this activity.

From this observation, we concluded that students could see the double of the *fives* but not double of other numbers in a math rack. The 5 structures are easy to recognize since they are differed by the color. These color differences did not support students understanding of the idea of double. Therefore, we think the math rack is not the right tool to introduce double strategy for students, and that implies we should think of other tool to replace the math rack in our next HLT.

Doing the decomposition to 10 however was not as difficult as the double. We assumed that this was because students just did the activity of doing decomposition to 10 with the math rack a day before. The groups of 10 structure in the math rack has enabled students to determine the final result just by looking at the beads in the bottom bar.

5.7. Analysis throughout all lessons

In this analysis, we looked at all lessons and searched for connections between them. Special attention was given to students' learning trajectory throughout those lessons as we wanted to see whether the activities have successfully helped students to move from a concrete level to a formal level.

In the HLT, we designed the candy packing activity (Lesson 1) to evoke students' awareness of structures and the result of the experiment showed that this activity was satisfactory. We found that the candy packing was a good context since students like candies and the packing problem has stimulated them to mathematize. In this activity students could use the structures, namely the groups of 5 and groups of 10 to help them move from counting all to counting by grouping. Students were evoked to employ the structures in the candy packing to do counting by grouping or by conceptual subitizing.

In the second lesson, we introduced the double sums through the double song. The song was followed by grouping activity in which we expected students would make a connection between the lyrics of the song and the double structures in their group. After that, students worked on the coloring worksheet, where they had to construct double structures and tell the quantity of a group of objects. Students' work indicated that students did not immediately relate the double song to the double structures. The reason for this might be because in the grouping activity students did not grasp the structures of the group. Students were the members of the group which made it difficult for them to see the structures as a composed of smaller groups since they were unable to participate and observe at the same time. On the contrary, in the worksheet they were not in the group which enabled them to observe the structures. Thus we concluded that students have not made the connection between the song and the double structures.

In lesson 3 we found out that the connection between the candy packing (lesson 1) and the egg box (lesson 3) was missing. We should have made that connection clear for students. Nevertheless, the activity with the egg box has successfully brought students from a concrete level of thinking to a formal level of thinking. We also found that this egg box has enabled the smooth transition of the *model of* to *model for*. First, students worked with the real egg box and then it was replaced by a schematized egg box. At this level, the schematized egg box served as the *model of* the real egg box, as students still relate schematized to the real egg box. As students used the structures in the schematized egg box to determine the number of the eggs, this schematized egg box changed its role to a *model for* the group of 5, group of 10 and double strategy.

In the 'The sum I know' worksheet we found that students did not immediately use the double song as a reference for double sums. This finding reinforced our conclusion that there is a missing link between the double song and double structures. In our next HLT we need to add more activities to bridge that missing link. In addition, we also found that the 'The sum I know' worksheet did not work as we expected in our HLT. We had conjectured that students would discover the number relation within the worksheet but this did not happen because of the absence of classroom discussions.

In lesson 4, students started to learn about the decomposition to 10 strategy. The egg box structures (lesson 3) and the first letter trick (lesson 2) were used to introduce the friends of 10. We found that both the egg box structures and the first letter trick were powerful to help students finding the friends of 10. This is a good start for doing the decomposition to 10 strategy. After that, students learned to decompose the other addend with the help of the math rack. We conjectured that the structures in a math rack would allow students to see the splitting of the other add end. Therefore, in lesson 5, this tool was used by some of the students.

However, the observations showed that adding 2 numbers by using a math rack was a complicated concept for students. Students were used to do addition up to 20 by counting on with fingers. The math rack was a new way for them, and thus at the beginning the rack did not serve as a tool to represent their thinking, but only to represent the result of the addition. After doing more activities with the math rack we finally saw student's improvement in using the math rack. Gradually they developed an understanding of the double and friends of 10 with the math rack.

During the observations in lesson 6, we found that when using a double strategy, students were more likely to double the groups of 5 in the math rack. This was a natural way for students as the groups of 5 were clearly visible by the color. This finding has shown that students have discovered an informal strategy that is doubling the fives.

Working with the decomposition to 10 was more complicated than the doubling, since decomposition to 10 requires more steps. As we have discussed earlier that to do the decomposition to 10, first students need to find the friends of ten. To enable students in finding the friends of 10, we used the egg box, finding the friend game and the first letter trick. From these approaches, students have been able to find the friend of a number. Moving up, the next step of decomposition to 10 was decomposing the other addend. We use the math rack to help students understand this concept. With the rack they could see how many more should be added after they got the 10. The observations have shown that the egg box and the math rack have enabled students to conceive the complicated concept of decomposition to 10.

Our other main concerns were the socio norms and the socio-mathematical norms. During the experimental phase, we have observed that this classroom has developed an open socio mathematical norm in which students were allowed to use different strategies. However, discussing and comparing strategies was not yet an established socio mathematical norm. This circumstance has disadvantaged the low achiever students as they were not encouraged to move form a concrete level to a formal level.

We found students' cooperative group work as an evidence of the socio norms. In the group work, each student contributed by giving ideas and sharing the works. The socio norms still need to be developed on a bigger scale such as in classroom discussions. As we observed, in most lessons, the classroom has not maximized the use of a discussion. A discussion should be an important moment for students in which they could learn from others. As a result of a discussion, students would get a classroom agreement for example that certain counting strategies are better than others.

5. 8. Final assessment

At the end of the teaching-learning experiment, we conducted an assessment to see whether our activities have resulted in any positive effects for the students. The assessment took form in a written test of 15 problems (see appendix). The problems were about structuring and addition strategies. In this section, we describe the analysis of students' written answers of the test. When designing the assessment, we made conjectures of how students would choose their strategy for solving the problems. In this analysis, we will compare students' actual strategy choices and our initial conjectures. We analyzed the test result of 35 students.

First, we made a frequency analysis based on the strategy choices that students made. As a preparation for the frequency analysis, we made a list of the expected strategy chosen by the students. Then we looked through each students' work. We recorded the strategies used by the students in the frequency analysis table. From this frequency analysis table, we get information of what strategy was chosen by the majority of the student and then we discussed the students' thinking process that underpinned their answers. From that, we got information whether the problems are appropriate enough to test students' thinking process.

Problem 1, 2 and 3 asked the students to do some structuring. These problems were similar to the candy packing activity except in a different context. The use of different contexts should enable us to see whether students were able to generalize their knowledge by applying it in other context. In problem 4, students were asked to decompose a number, that is 17 into some smaller numbers. In this problem, we wanted to see what strategies are used by the students when decomposing a number. We gave them the option of numbers which allowed them to use either doubles or friends of 10.

Problem 5 was taken from the "The sum I know" worksheet. The intention of this problem was for students to use the double strategy in solving almost double problems. In problem 6 to 9, students were asked to work with math rack structures. We aimed at knowing if students were able to use the double or the decomposition strategy when adding two numbers by using a math rack. Therefore, we deliberately gave the picture of the math rack to guide students to use either the double strategy or the decomposition to 10 strategy.

In problem 10 to 15, we gave students formal addition problems. Here, students had the freedom to choose any strategies they like, since our aim was to see what strategy students used to solve addition up to 20 problems. We expected students would use the double strategy or decomposition to 10 strategy.

In the following section, we discuss students' work in detail

5.8.1. Problem 1

In this problem, students were given a school bus context in which they had to arrange where the passengers should sit so that the driver can count the number of the passenger easily. In this problem, we aimed to see students' structuring awareness which was indicated by the way they put the passenger in the bus. We expected students would put the passengers in a structured configuration that allows them to do conceptual subitizing.

The frequency analysis showed that 33 out of 35 students were able to put the passengers in a structured arrangement. However, we realized that this did not indicate students' ability to structure since the seat given was already structured. We observed, that most students added the number of the passengers in each bus stop and after they found the total number, they started to mark the seats. Therefore this problem did not ask the students to structure by themselves. We propose, it will be better if students do the arrangement themselves, therefore for the next cycle, the context should be changed. For example, students can be asked to arrange the seats for guests in a party.

5.8.2. Problem 2

In problem 2, students were tested whether they knew the double structure or not by making a choice of which chocolate bar can be equally divided. The context given was about a fair sharing situation. Students had to choose which from the 3 chocolate bars given is divisible by 2. The first chocolate bar (Cokelat Enak) consisted of 12 pieces of chocolate in a 3 by 4 configuration, the second chocolate (Cokelat Mantap) was in a 3 by 3 configuration and the third chocolate (Cokelat Manis) was in a 2 by 6 configuration. We conjectured the students would choose the third chocolate since the double structure was clearly visible.

The students' work showed that most of the students chose both cokelat manis and cokelat mantap because they consist of the same number of pieces. This was a natural answer from students which we did not anticipate before. This has shown that the problem was designed from an actor's point of view. For our next cycle, we suggest to use a different context.

5.8.3. Problem 3

In this problem, we used the packing context again where students were asked to make boxes of 12 and 20 strawberries. This context was similar to the candy packing context, only this time, students did not have the concrete objects to work with. They had to make a drawing of their strawberry boxes. We conjectured that students would use groups of 5, groups of 10, or double structures. More precisely, for 20 strawberries students might use groups of 5 or groups of 10, and for 12 strawberries, we expected some students would also use double structures.

The students' work showed that 16 students used groups of 5 structure and 18 students used groups of 10 structure. For example 20 strawberries were arranged in a box of 4 by 5 structure, while 12 strawberries were arranged in a 10 + 2 structures. We did not find students' work that showed 12 strawberries in 2 by 6 structures.

This finding indicated that the students were greatly influenced by the candy packing activity. The double activity with coloring seemed not too have much effect on the students. This means that students learn better in a physical activity that supports the learning of the intended mathematical concept.

5.8.4. Problem 4

In this problem, students were asked to decompose a number, 17. We gave students 4 numbers to use; 7, 6, 5 and 4. These options allowed students to use friends of 10 (i.e., 17 = 6 + 4 + 7), double (i.e., 17 = 6 + 6 + 5) or group of 5 (i.e., 17 = 5 + 5 + 7). The aim of this problem was to see the structures used by the students.

The students' work displayed that 25 of 35 students use the friends of 10, which might indicate that students were more familiar with the friends of 10. However, we realized that the type of question also played an important role in students' choice of answers. In this context of saving bonus points, (i.e., where students collect bonus points to trade with items) students might tend to trade the points with various items instead of getting more of the same item, therefore the double strategy was not chosen by the students.

5.8.5. Problem 5

In this problem, we aimed at testing students' ability to use double structures to solve almost double problems. We took a small part of the "The sum I know worksheet" to give students a clue of the expected strategy, that is the double strategy. We conjectured that by using part of the "The sum I know" worksheet, students would be guided to use the double structures.

However, the result indicated that not many students were stimulated to use double structures. We found that 13 students used the double strategy and 13 other students used the decomposition to 10 strategy. They did not automatically use the double strategy even though the problem clearly promoted them to do so. Students might have seen every problem as an individual problem instead of relating them to each other. This implies that some students might have not fully grasped the number relation concept in the "The sum I know" worksheet.

5.8.6. Problem 6 to 9

Students were presented the structures in a math rack by problem 6 to 9. In these problems, students had the pictures of the math rack in which the numbers being added were presented. The pictures were aimed to help the students choose the easy strategy. In problem 6 and 7 we conjectured that students might use a double strategy, while in problem 8 and 9 students might use decomposition to 10. However, for problem 6 and 7, we found that students preferred to work with the groups of 5. This has also been seen in the 6th lesson, in which the students employed the groups of 5 structure instead of the double.

In problem 8 and 9, most students used decomposition to 10. They did not show any difficulties in splitting the other addend. This might have been supported by the first letter trick which helped students to decompose the other addend.

5.8.7. Problem 10 to 15

The last 5 problems were formal addition problems that did not suggest the students to choose any particular strategy. They had the freedom to choose the strategy they like. We conjectured many students would use decomposition to 10, and we did not expect students would use doubles, since the problems were not almost double.

As we have expected, the majority of the students used decomposition to 10 to solve these problems. However, students did not use the groups of 5 strategy as seen in the previous problems. This then raised a question; do students used the groups of 5 strategy only when stimulated by a math rack? Looking at students' mathematical history in grade 1 where they learned about the friends of 10, this might answer the question. Students were accustomed to the friends of 10 thus, they used decomposition to 10 frequently. On the contrary, they just knew the group of 5 from the math rack activity, thus they were not too familiar with it.

Summing up, we concluded that the choice of problems and the numbers in the problems greatly determine students' way of thinking. Thus problems should be carefully chosen to guarantee the validity of the test. Some problems in this test reflected students understanding of addition up to 20 which also correlated to the activities. In problem 3, we saw that the strawberry packing was greatly influenced by the candy packing activities. We could conclude that students have applied what they learned in the candy context in different context.

The "The sum I know" worksheet did not immediately promoted students to use double strategy for solving almost double sums. This implies that students haven't grasped the number relations, as they still see each addition individually instead of relating it to other additions.

Students' tendency to use groups of 5 with a math rack was found in the lesson and in the assessment. In problem 6 and 7 where the structures in the math rack led students to use the group of 5 instead of double. This finding was also found in lesson 6. From this result, we suggest to improve the activities on the double structure as we haven't seen any strong evidence that students have understood the double strategy.

Chapter 6

Conclusion and Discussion

The goal of this research is to develop classroom activities that support students' learning of abbreviated strategies of addition up to 20. The contribution of this research will be adding a local instruction theory of teaching addition up to 20 in Indonesia. The local instructional theory provides knowledge about classroom activities that support students in abbreviating strategies of addition up to 20.

We have posed a general research question "how can structures support students' learning of abbreviated strategies for addition up to 20?" which then is specified into more detailed sub-research questions. In this chapter, first we answer the research questions based on our retrospective analysis. Then we reflect on some important issues in this study and elaborate recommendations for further studies and improvement of mathematics education in Indonesia.

6. 1. Answer to research questions

6.1.1. Question 1: What kinds of structures of amounts up to 20 are suitable in the Indonesian context?

To answer this question, we looked at the retrospective analysis of the activities during the experimental phase of the research. In the candy packing activity, students were asked to arrange the candies in such a way that they would be easy to count. Students naturally chose group of 5 or group of 10. They put the candies in a line structure of 5 or 10 candies. Based on this observation, we concluded that line structures may be the simplest structure for children. This implies that line structures can be used to introduce structures for young children.

The structures of the egg box promote students to reason by groups of 5, groups of 10 or doubles. In chapter 5, we have described the transition of students reasoning from concrete objects to a more formal mathematics. At first, the egg box served as a concrete object which is embedded in the situational problem. After that, the egg box was replaced by the schematized picture. Here, the picture was a representation of the real egg box. In accordance with Gravemeijer et al., (2000) in this moment, the schematized picture was the *model of* the egg box. Gradually, when students no longer associated the schematized picture with the egg box, then it transformed in to a *model for* formal reasoning. To be more specific, the egg box allows students to reason by using groups of 5, groups of 10 and double structure.

Beside the egg box, we also used the math rack to help students employ structure in their mathematical reasoning. The math rack has allowed students to reason using groups of 5 and groups of 10 structure based on the colors of the beads and the groupings on the two bars. However, this research has shown that the math rack did not support students' reasoning on the double structures. Color differences have driven students to only look at the groups of 5, not at the double structure. This implies that the egg box structure is more powerful for introducing double structure.

However, in most places in Indonesia, the egg box is not a good context since it's not widely used in Indonesia. Therefore, we recommend teachers and educators to find other concrete objects that have the 5 structures, 10 structures or double structures such as an ice tray.

6.1.2. Question 2: How does the role of structures evolve when students learn to abbreviate the strategies of addition up to 20?

To answer this question, we looked at the sequence of learning activities and investigate what role the structures serve in each sequence of students' learning. After that we can conclude how the role evolves during the activities.

This research has hypothesized that students will not employ structures unless they realize the benefit of structuring for counting and arithmetic. Therefore in lesson 1, we conducted an activity to evoke students' awareness of structure in which students learn to recognize and construct structure. Through this activity, students realized the need of structuring and start to recognize and use structure for their mathematical reasoning. In this lesson, students have set up a new paradigm of thinking. They started to use structures such as groups of 5, groups of 10 and doubles in their thinking process. We consider this as a critical moment for students, since they will see more structures in the whole body of mathematics.

The awareness of structures has given a base for students to employ structures for further counting and adding. After students were able to recognize the structures of a collection of objects, then they could use the structures in the reasoning about their counting. In lesson 2, we found evidence that students started to use the structures for conceptual subitizing and counting by grouping. This finding supports previous work from Benoit, Lehalle, and Jouen (2004) that children rely on the presentation and look for the patterns in a configuration to determine the quantity of a collection of objects. They can see that a collection of objects is made of smaller collections. This has shown their ability to decompose a group of objects into smaller groups of objects. On the other hand, students were also able to do the reverse, which is composing a group of objects from some smaller groups of objects.

In line with Gravemeijer, et al., (2000), this research has shown that structures have triggered students to employ the *model of* a real situation for solving problem. The schematized picture of the egg box structure allowed students to relate to the concrete egg box. Gradually, students were able to see the schematized picture separately from the real object and used it as a *model for* counting or adding by groups of 5, groups of 10 or double strategy. Even though in this research students did not invent the model themselves, they have shown an ability to use the model as a bridge to move from concrete objects to more formal mathematical thinking.

The research showed that the math rack has helped students represent their thinking. The structures in the math rack promoted students to reason by groups of 5 and groups of 10 structure. For example, when solving 8 + 7 with a math rack, students put 8 beads in the upper bar and 7 beads in the bottom bar. They could see that 8 beads were made of 5 red beads and 3 white beads; 7 beads are made of 5 red beads and 2 white beads. Then they group the beads based on the color. They have 10 red beads and 5 white beads which altogether made 15. The structures in the math rack has promoted student to decompose and recompose numbers (i.e., $8 + 7 = \ldots$; $8 = 5 + 3$; $7 = 5 + 2$; $(5 + 3) + (5 + 2) = (5 + 5) + (3 + 2) = 10 + 5 = 15$).

The math rack has allowed students to represent a number in different ways. This can be a start for students to learn about number relations. This understanding is the base knowledge of students' further mathematical development such as associative property of addition.

To be able to do the decomposition to 10, students have to do 4 steps as mentioned in chapter 2. In line with Van Eerde (1996), this research has shown that the first letter trick and the egg box structures have helped students retrieve the friends of 10 which is one of the steps of doing decomposition to 10. The next step, the decomposition of the other addend was supported by the structures of the math rack. Students could represent the decomposition from the movement of the beads in the math rack, and finally they can perform all the steps in the addition strategies.

However, we found that the math rack did not support students' learning of the double strategy. The color differences in the math rack have promoted students to use group of 5 instead of the double strategy.

In short, this research has shown that the structures of groups of 5, groups of 10 and double play a central role in students' learning trajectory. The structures served as the basic knowledge that underpins the abbreviations in the counting and in the addition strategies. By recognizing the structures in a collection of objects students can move from counting all to counting by grouping, to conceptual subitizing. After that, the *model of* and *model for* is used to bridge the transition between concrete objects to formal mathematics. With the model, students can employ the structure to support their thinking. Next, we focused on using structure to help students learn the abbreviated strategies of addition up to 20. In order to do that, we used the structures in a math rack. We found that the math rack has successfully support students' thinking of doing the decomposition to 10 strategy and the groups of 5 strategy.

6.2. Reflection on the important issues

This research has contribute to the addition of local instructional theory in which provides information about classroom activities of supporting students' learning of abbreviated strategies of addition up to 20. Throughout this research we have tried out activities to foster students learning about strategies of addition up to 20. In this section, we highlight some important issues based on the findings of this research.

6.2.1. Realistic Mathematics Education

This research has shown students learning about strategies of addition up to 20 by using structures. In this research, some ideas and concepts from RME theory has underpinned the design of the activities. The context used was about candy packing and we found that this is a good context that has allowed students to structure and to mathematize. They first started by using concrete objects namely candies and they were stimulated to construct structures for a better way of counting. In this activity, not only have students become aware of structure, but also they have mathematized by breaking and grouping in search of the best arrangement. This context has evoked students' awareness of structure, which is an important foundation for further learning.

In the activities, we have seen how the *model of* transformed into a *model for*. This transformation is another important learning moment for students where they can use the model to move from concrete objects to a more formal mathematics. However, we realize that the students did not model the situation themselves but it was given by the teacher. Therefore, in our next HLT, we need to add more activities that allow students to model situations.

6.2.2. Classroom discussion: new socio norms

In this research the classroom discussions did not run as effective as we had expected. The classroom we worked with has established an open learning environment where each member of the class had the freedom to express their ideas and share them with other members of the class. However, in this classroom, the teacher and the students were still struggling in developing a constructive discussion. With a large number of students in a class, in our case there were 37 students, the teacher's role in orchestrating a discussion is not easy. Only few students were engaged in the discussion while many others were busy doing something else.

We found that in this class, all students were allowed to use their own strategies, from counting to the most sophisticated one. This could have been a constructive learning environment, if the class had developed a socio-mathematical norm in which students discuss and compare strategies, and are promoted to use abbreviated strategies. The absence of these discussions and socio-mathematical norms did not support low achieving students to move to a more formal strategy.

Teacher and students can learn to develop these socio-mathematical norms through a classroom discussion. Moreover, we suggest, changing students' seating position (e.g., students sit in a circle) might help to improve the quality of a discussion. However, this is not feasible due to the classroom setting which doesn't have enough room for a big number of students to sit in a circle. Another possible solution is by having discussions within small groups. There are some advantages of students working in a small group such as, it allows students to give more contributions and also minimize the number of students' solutions. After that, each group may present their work in front of the class, and all students participate in comparing and choosing the best strategies.

6.2.3. The role of the teacher

During the experiment, we have also observed the role of the teacher in students' learning trajectory. In this section, we elaborate the role of the teacher throughout the series of the activities.

We have discussed about context as one of the five tenets of RME (chapter 2), the role of the teacher is very essential in order to connect the context and the mathematical learning. In this research, the teacher has developed the context, through story telling or posing problems in such a way that students are engaged and stimulated to mathematize. She has also shown a good performance in keeping the consistency of using structures all the time. She has always encouraged the students to employ structures to support thinking process in every lesson. She has made a good connection between the visual representation of the structures and the formal mathematical notations so that the transition from concrete to abstract can be seen by the students.

However, during the experiment, we missed some learning moments due to the absence of classroom discussions. Therefore, the teacher's role of conducting a constructive classroom discussion is very important. The teacher could have maximized the classroom discussions if she had given more opportunity for students to explain their strategies, justify solutions, and discuss the best strategy. In this way, she would promote the development of the classroom socio-mathematical norms.

6.3. Recommendations

In this section, we give recommendations concerning realistic mathematics education in Indonesia and using structures in mathematics lesson. These recommendations are made based on the finding of this research.

6.3.1. Realistic Mathematics Education in Indonesia

Realistic mathematics education (RME) can contribute to developing a traditional teaching in Indonesia to a more progressive learning. In our research, RME has underpinned the classroom activities and we have seen how students learned better in such an environment. The use of context, in this case we used the candy packing context, has stimulated students to think of a way of solving problems. With a good context, mathematics is meaningful so that it makes sense for them. A context also should allow students to mathematize by creatively inventing new strategies, in that way, students can construct their own understanding.

The emergence of models supports students' transition from concrete situational problems to more formal and general mathematics. The model serves as the stepping stone for students. With a *model of* students relate the concrete objects to the model, and with a *model for* students can use the model to represent and support their thinking.

In RME classrooms, the contributions from the students are highly promoted. Students learn to share and listen to each other's idea through a discussion where strategies are discussed and compared to determine which ones are more sophisticated. In a discussion, students can learn from their peers and the collaborative development of knowledge among students can be made possible.

During the research, we found that the classroom we worked with was still struggling in establishing socio norms and socio-mathematical norms. Nevertheless, a good start has been made as this class has developed an open learning atmosphere where students are allowed to use their own strategy. In this classroom students have freedom to use different strategy, but they are not promoted to discuss and choose the best strategy. This condition does not support the mathematical development of low achiever students because they are still allowed to use counting and not stimulated to use other more sophisticated strategies. More efforts are still needed to continue the development of the socio-mathematical norms, where students are aware and have the ability to choose the best strategy. We realize that it takes time for students to develop such norm, but it is important to keep on nurturing the students with constructive learning attitudes.

6.3.2. Using structures in mathematics lesson

From this research we have contributed to a local instructional theory about classroom activities that support students learning of addition up to 20 by using structures. In this section, we recommend teachers, educators and public policy makers to employ structures in school mathematics.

This research has shown the use of structures in teaching students how to abbreviate addition strategies. We also think that structures can be used in many other mathematical domains such as geometry and statistics. By structuring, students are stimulated to find patterns, make connections, group and decompose elements to find a better strategy of solving mathematical problems.

We have shown that structuring allows students to mathematize, that is by recognizing patterns and regularities and use the patterns for further thinking. Structures in an egg box and math rack has supported students to construct double and decomposition to 10 strategy of solving addition up to 20 problems. Many researches have shown the advantages of structuring in other early mathematics domains. Van Nes (2007) used spatial structures for developing students' number sense. Mulligan, et al (2004) showed that students' ability to structure has a positive correlation with their mathematical achievement. Thus we recommend public policy maker to include structuring in school mathematics curriculum in Indonesia. The breaking and grouping, finding patterns in a collection is an important informal mathematical skills which haven't got much attention in school mathematics. By structuring, students learn to look at a problem from different perspectives that allow them to discover a creative solving strategy. Nevertheless, in Indonesian school, students are not used to this creative way of thinking since they are always taught the standard procedure or the algorithm of solving problems.

6.3.3. Further studies

This research has found differences in students' ability in recognizing and using structures. We could hypothesize some levels in the development of using structures for doing addition up to 20.

1. *Non structural*: Students who do not recognize structure and keep on rely on counting all strategy.
2. *Informal structural*: Students who can recognize structure of small group by perceptual subitizing, and then use it for conceptual subitizing, but can not do counting by grouping due to their inability of basic addition. Therefore, at some point they still use counting all. For example: when solving 8 + 6, a student could show an *eight* in the upper bar of a math rack by moving 5 beads at once, and then 3 more beads at once. After that he showed *six* in the bottom bar by moving 5 beads at once and one more bead after that. But when determining the total number of beads, this student still used counting all strategy. At this level, students are able

recognize the structures but unable to employ the structure in their formal mathematical thinking.

3. *Formal structural*: Students who can recognize structure and use structure for their counting reasoning.

These findings have raised some new questions such as, how do students differ in their ability of structuring? What basic knowledge underpins students' ability to structure? How to support students to structure better? Further research is needed to answer those questions.

6.4. Reflection on the learning process of the researcher

There are so many things I have learned from this research. The lesson I've learned were not only about conducting a good research but also about many other aspects in life such as networking and cooperation in a team work. In this last section, I would like to reflect my learning process during the research.

Doing a research with young students has taught me to expect the unexpected. Students' actual learning do not always meet the expectation in the HLT. There are so many things I missed to anticipate. I learned that having more discussion about my activities with experienced teachers or researcher to have a better anticipation for my future research. Other thing that I can do is by reading more research reports and studying how the researchers conducted their research.

It is important to keep a good communication with the teachers and convince the teachers, that the research could benefit their professional development. In that way, they will be willing to learn while participating in the research. I learned that in this research, the teachers and the researchers worked as a team. The trust in the team needs to be developed from the beginning for example by inviting the teachers in the discussion of the activities and appreciating their inputs for the research.

Design research requires a good data organization. During this research, I have collected many data; therefore a good organization is needed to make the analysis more efficient. Having all data registered is very important in order to keep track of the data.

Last but not least, I realize that this research is far from perfection as there are many things need to be improved. I learned that writing an honest research report is important, because as a researcher, we can criticize our own work, learn from our mistakes and let others to learn from that too. I also learn to accept critics from others as a positive input for the improvement of my next research.

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