Computational estimation in grade four and five: Design research in Indonesia

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Preface

When I studied for my bachelor degree in mathematics education, my interest was mathematics education in secondary school. However, after I graduated in 2004, I became involved in the PMRI project as an observer of the implementation of realistic mathematics education (RME) in primary schools in Indonesia. At that time, I had no idea that through this project I would have a chance to continue my study at the Freudenthal Institute, Utrecht University, the Netherlands, to get a master degree in "*research and development in mathematics education*". This master program had caught my interest since I would be a researcher in mathematics education if I completed the program.

I think the first half year of my study was not easy. This is because I had to adapt to new social cultures and a high quality of academic cultures which demand a very good proficiency in English. Later on, after struggling this first half year, I enjoyed completing the study here, at the Freudenthal Institute. Through this wonderful institute I learned many things both in terms of academic and research cultures.

As a new prospective researcher I had to conduct a research to complete my study. The research should be conducted at primary schools to align with the PMRI project. I chose "estimation" as the topic of my research. The reasons of my choice were the following. First, this topic is interesting because it is used a lot in our daily life but interestingly it is so little taught. Second, I think estimation can be integrated in most of the school mathematics curriculum, where through this topic the intertwinements between mathematical topics are apparent. And third, I think estimation can be an interesting research topic for secondary education—fitting my interest when I was working for my bachelor degree.

In the early stage of my research I was supervised by Nisa Figueiredo. She patiently guided me how to prepare the research. One thing that I will always remember from her is that I have to focus on "small but deep" concerns for the research investigation. Therefore, for the supervision I would like to express my gratitude to her.

During the research in Indonesia, I got much help from many people. I would like to express my gratitude to Bu Upi Piasih, as the teacher in the teaching experiment. From her I learned a lot on how to interact with students, how to manage classroom situations. I also would like to thank Bu Lastri Sulastri, her class was used during the first research period. Thanks to Bu Tia who assisted me to use a video camera during the teaching experiment. Thanks to Bu Eni, Bu Nila, Pak Toto, and other school teachers for their kindness during the research. I would also like to thank the school principal, Pak Sholeh, who allowed me to conduct the research in his school. Of course, I would also like to express my gratitude to my Indonesian supervisors: Pak Dian Armanto who flied from a very far city to my place just for supervising me, his supervision was helpful in focusing my observation in the teaching experiment; and Pak Turmudi who supervised me in many ways: he is my supervisor, teacher, and also a friend. And I also thank my colleagues at the Department of Mathematics Education, Indonesia University of Education: Pak Russefendi, Bu Utari, Pak Darhim, Pak Kosim, Pak Yaya, Pak Yozua, Pak Wahyudin, Pak Didi, Pak Tatang, Pak Dadan, Bu Siti, Bu Dian, Bu Dewi, Bu Entit, Bu Ellah, Pak Kusnandi, Pak Endang Dedy, Pak Endang Mulyana, Pak Endang Cahya, Pak Rahmat, Pak Rizky, Pak Cece, Bu Kartika, and others who encourage me in succeeding my study.

Through the research in Indonesia perhaps my interest in mathematics education has changed gradually. I realized that there are still large problems in mathematics education at primary schools level, and this is as interesting as secondary mathematics education problems or even more.

With pleasure, I would like to express my very deep gratitude to my recent supervisor, Arthur Bakker. He carefully supervised me during the final stage of the research: writing this thesis. From his supervision I learned many things: I learned how to manage large data and present them in a very short and representative way, present ideas precisely into writing, analyzing data (not only presenting facts or telling stories, but also giving reasons and also drawing conclusions), and understand the design research as a research method in developing mathematics education better than previously. By his supervision, I

think and feel my dream as mentioned before, namely to become a researcher in mathematics education, will come true in the near future.

During two years of my study, I got much help from the Freudenthal Institute members. I would like to express my gratitude to Betty Heijman who managed my study matters here, Maarten Dolk as the PMRI project leader from the Freudenthal Institute, Jaap den Hertog as the coordinator of the master program, Mark Uwland who corrected my written English, Liesbeth Walther who helped me on housing matters, and others: Jan van Maanen, Henk van der Kooij, Dolly van Eerde, Aad Goddijn, Martin Kindt, Marja van den Heuvel-Panhuizen, Monica Wijers, Marjolijn, Michiel, Corine, Frans, Paul, Ronald, Wim, Ellen, Ank, Bart, Wil, Mariozee, etc. Their kindness in helping me in many ways will always be remembered.

I would also like to thank Kees Hoogland from APS that provides the scholarship during my study. I also thank the Indonesian PMRI members: Pak Sembiring, Pak Zulkardi, Pak Sutarto, Pak Pontas, and Mba Marta for their support in this program. I also thank the other six Indonesian master students: Ari, Meli, Neni, Novi, Puspita, and Rose who always stayed together as friends during my study. I would like also to express my deep gratitude to Mas Yusuf Setiyono; his very conducive house where I have stayed for one year made me enjoy my study in the Netherlands. Therefore I will always remember his kindness. I would also thank the Indonesian community in Utrecht: Mas Untung and family, Bang Andi and family, Mas Pardi and family, Kang Ari and family, Mas Agus and family, Mas Seno and family, Mas Bambang, and others that made me feel like I was in my lovely country.

Last but not least, I am very grateful to my parents, my sisters, and my brothers who always motivate and pray for me when I was studying in this country, a very far country like a land that only exists in a dream. That is why I dedicate this master thesis to them.

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1 Introduction and research questions

Mathematics is indispensable in our daily life. From waking up in the morning to sleeping at night, we always use mathematics, be it perhaps implicitly. It is no wonder that Freudenthal—a mathematician as well as a mathematics educator from the Netherlands—said that mathematics should be seen as a human activity (Freudenthal, 1991).

One of the branches of mathematics used most to solve our daily life problems is arithmetic. Everyday we use four basic skills in arithmetic: addition, subtraction, multiplication, or division. One calculation form which is frequently used is computational estimation. For instance, when we are in the supermarket, we should be able to calculate the prices—without using paper and pencil or calculator but using mental calculation—of goods that we want to buy before going to the supermarket's cashier, whether our money is enough or not, whether it is suitable with planning or not. Therefore, computational estimation should be learned at school.

Computational estimation, as a basic skill in mathematics (Reys, Rybolt, Bestgen, & Wyatt, 1982), is acknowledged by many educators as an essential skill which should be mastered by students (Rubenstein, 1985). Many mathematics educators suggested that estimation is commonly used more than exact calculation in daily life (Rubenstein, 1985). For example, consider the following estimation problem:

> **Example 1.1:** *The price of* 1 *kg of cabbage is* Rp 1,675. *If you have* Rp 10,000*, is it enough to buy* 5 *kg of cabbage?*

Instead of doing the exact calculation 5 x Rp 1,675, it is sufficient to do, for example, 5 x Rp 2,000. Thus, we can easily conclude that the money is enough to buy 5 kg of cabbage.

Another example of the importance of computational ability for students is that students will be able to check reasonableness of computational results, for instance calculation by calculators (Rubenstein, 1985). As an example, if a student wants to enter 213 x 15 into a calculator, the answer 325 appears on the display.

Then using estimation ability, it can be shown that the answer is incorrect because, for instance, 200 x 15 = 3000 is more than 325.

In addition, according to Van den Heuvel-Panhuizen (2001), estimation has a didactical function for learning to calculate exactly. Doing estimation beforehand can help to master mental calculation strategies for doing algorithms of mental arithmetic. For example, with the following problem pairs: (1) 3 x 97 ≈… and (2) 3 x 97 = … The problem (1) can elicit an understanding that the problem (2) can be calculated by (3 x 100) – (3 x 3).

However, despite its importance, estimation is perhaps the most neglected skill area in mathematics curricula (Reys, Bestgen, Rybolt, & Wyatt, 1982), even over the world (Reys, Reys, & Penafiel, 1991). For example, in the Indonesian mathematics curriculum, estimation is introduced in the grade four primary school students as a subtopic of whole numbers and it is extended in grade five (Depdiknas, 2006). There is no clear-cut reason why estimation is so little taught. It could be because it is difficult either to teach or to test (Reys, Rybolt, Bestgen, & Wyatt, 1982). Or, it could be because people assume that if one can calculate then one can estimate automatically.

According to Trafton (1986), most students are uncomfortable with estimation. There are several possible reasons. Students are not sure why they need to do estimation; They find that estimation frequently requires paper and pencil and a great deal of time to produce an estimate; They are not convinced if they solve estimation problems by estimation strategies, to make sure they frequently work out the exact answer on paper first to get an estimate. In addition, students view estimation as invalid mathematics, where they regard that mathematics deals only with exact answers. They often ask why they can not find the 'real answer' directly.

Moreover, (Van den Heuvel-Panhuizen, 2001), although the calculation work of estimation is much easier, estimation problems actually turn out to be very difficult for many students. This can be seen from the results of the 1997 PPON survey of arithmetic skills in the Netherlands: only one third of the students in grade 6 could estimate the answer for a problem in which eight winners split a prize of 6327.75 euro. This could be because, when faced with estimation problems, most of the students do not estimate at all even if the problem requests an estimate. In this case, one problem might be that students do not immediately see that 6400 can easily be divided by 8 and is close to 6327.27. These results reflect the position that has long been held by estimation in arithmetic education. There is a long tradition of exact calculation. In particular, learning to calculate was—and frequently still is—involved exaggerated in Indonesia is done with the careful performance of operations.

Because of the above issues, we conducted research on computational estimation with the aims: (1) *to investigate students' strategies in solving estimation problems*; and (2) *to gain insight into how students can be stimulated to use estimation strategies instead of using exact calculation in solving estimation problems.* In to the light of these aims, we conducted a design research with the following research questions:

1. *What strategies do students use to solve estimation problems?*
2. *What are students' difficulties in solving estimation problems?*
3. *What kind of problems invite students to use estimation?*
4. *What kind of learning-teaching situations invite students to use estimation?*

2 Theoretical framework

We begin this chapter with a literature review of computational estimation which is used both for a basis in designing the research instruments and for explaining the research results. Next we describe realistic mathematics education (RME) as a didactical and pedagogical theory for designing either the research instruments or the learning-teaching situation in this study.

2.1 Computational estimation

What is computational estimation? There are several definitions of this term. Here we present two that we find most relevant in the context of grade 4 or grade 5. First, *computational estimation* is the process of simplifying an arithmetic problem using some set of rules or procedures to produce an approximate but satisfactory answer through mental calculation (LeFevre, Greenham, Stephanie, & Waheed, 1993). And second, according to Dolma (2002), *computational estimation* is nothing more than quickly and reasonably developing an idea about the quantity of something without actually counting it. We synthesize these definitions into our own: computational estimation is the process of simplifying an arithmetic problem to find a satisfactory answer, without actually counting it, through mental calculation. Example 1.1 is an example of computational estimation problems. Other examples are as follows.

Example 2.1: *If the price of* 2 *bundles of Kangkung* (*a kind of green vegetables*) *is* Rp 3,750, *can you buy* 5 *bundles of Kangkung with* Rp 10,000?

Example 2.2: ***Local News,*** *"This afternoon, there are* 9998 *supporters of PERSIB Bandung who will go to Jakarta using* 19 *buses to support their team against PERSIJA Jakarta." What do you think, does the news make sense?*

Regarding the learning-teaching of computational estimation, Van den Heuvel-Panhuizen (2001) distinguished three types of questions—where the questions can take on all kinds of different forms—that are the driving force behind learning to estimate and which, moreover, are anchored in estimation as it occurs in daily life, namely: *(1) Are there enough? (2) Could this be correct?* and *(3) Approximately how much is it?* In the Netherlands, the first two types of questions are used in initial phases of learning of estimation because they are indirect questions, while the third type is used in next phases when students have sufficient experience in estimation because it is a direct question. Example 2.1 and Example 2.2 above use the first and second type of the questions respectively.

There are various types of estimation problems. The two most important types, according to Van den Heuvel-Panhuizen (2001), are as follows:

- Calculation with rounded off numbers: the intention is to find a global answer to a problem with complete data. See, for instance, Example 1.1 and 2.1.
- Calculation with estimated values: the intention is to find a global answer to a problem with incomplete or unavailable data. Example 2.2 is an available data problem. Example 2.3 below is an example of problems with incomplete data.

Example 2.3: *For each problem below*, w*hat could be the right answer:* A, B, *or* C?

28[redacted]
4[redacted] +
A. 62[redacted]
B. 7[redacted]
C. 557

9[redacted]2
489 --
A. 4[redacted]
B. 56[redacted]
C. 6[redacted]

79
3[redacted] x
[redacted]
[redacted]7 +
A. 2[redacted]8[redacted]
B. 226[redacted]
C. 33[redacted]3

What strategies do we use to solve computational estimation problems? We can identify three general cognitive processes among good estimators, namely the processes about how good estimators produce estimates, i.e., *reformulation*, *translation*, and *compensation* (Reys et al., 1982; Reys et al., 1991). The processes are actually strategies which are used to do estimation.

Reformulation is a process of changing numerical data to produce a more mentally manageable form. This process leaves the structure of the problem intact. *Reformulation* includes, for instance, *rounding* (e.g. 105 to 100), *front-end strategy* (e.g. 4112 + 5231 + 2925 as 4000 + 5000 + 2000), and *substitution* (e.g. (278 x 7)/15 as (280 x 7)/14).

Translation is a process of changing a mathematical structure of the problem to a more mentally manageable form. This process includes, for instance, *changing operation* (e.g. 13 + 15 + 19 as 3 x 15) and *making equivalent operations* (e.g. (268 x 7)/15 to be 268/2, so it is about 270/2).

Compensation is a process of adjusting an estimate to correct changes due to *reformulation* or *translation*. This process includes *final compensation* and *intermediate compensation*. *Final compensation* is adjusting an initial estimate to convey more closely the user's knowledge of the error introduced by the strategy employed (e.g. 8 x 1982 to 10 x 1982 = 19820, then finally since it is an overestimate it is compensated to be 19820 – 2 x 2000 = 15800, to compensate 2 x 1982). And the *intermediate compensation* is adjusting numerical values prior to their being operated on to systematically correct for errors (e.g. 35 x 55 to 40 x 50).

In our research in grade 4 and 5 we focus on: (1) an investigation of strategies used by students in solving estimation problems to understand what kind of cognitive processes they use; (2) an understanding of students' difficulties in solving estimation problems either to aid students in learning estimation or to design an instrument for estimation instructions that fits with students' thinking; (3) looking for problems that invite students to use estimation, where this would be a model for designing problems that support students' learning in estimation; and (4) in particular for grade 5, the research is also focused on a creation of learning-teaching situations that encourage students to use estimation as an exemplary of learning-teaching in estimation. To do these we use realistic mathematics education (RME) because it offers pedagogical and didactical both mathematical learning and instructional materials for learning-teaching instruction (Treffers, 1987; Gravemeijer, 1994; Bakker, 2004). In addition, the RME theory is appropriate with the learning of estimation—particularly in this research, where it is explored from experientially real life problems.

2.2 Realistic mathematics education

Realistic mathematics education (RME) is a theory of mathematics education which has been developed in the Netherlands since the 1970s and it has been extended there and also in other countries (De Lange, 1996). This theory emerged from design work and research in mathematics education in the Netherlands—especially at the Freudenthal Institute, Utrecht University.

RME is shaped by Freudenthal's view on mathematics (Freudenthal, 1991), namely: mathematics should always be meaningful to students and should be seen as a human activity. The term 'realistic' means that the problem situations should be 'experientially real' for students. This means the problem situations could be problems that can be encountered either in daily life or in abstract mathematical problems as long as the problems are meaningful for students.

There are five tenets of RME according to Treffers (1987) and Bakker (2004), which we summarize as follows:

a. *Phenomenological exploration or the use of meaningful contexts*. A rich and meaningful context or phenomenon, concrete or abstract, should be explored to support students in developing intuitive notions that can be the basis to build awareness, in particular, of the use of estimation.
b. *Using models and symbols for progressive mathematization.* A variety of context problems, models, schemas, diagrams, and symbols can support the development of progressive mathematization gradually from intuitive, informal, context-bound notions towards more formal mathematical concepts.
c. *Selfreliance: students' own constructions and strategies*. It is assumed that what students do in the learning processes, particularly in estimation, is meaningful to them. Students are given the freedom to come up with their own construction and strategies in solving estimation problems. Thus, these would constitute essential parts of instruction.
d. *Interactivity*. The learning process, especially on estimation, is part of an interactive instruction where individual work is combined with consulting fellow students, group discussion, class discussion, presentation of one's own strategies, evaluation of various strategies on various levels and explanation by the teacher. Hence, students can learn from each other either in groups or in whole-class discussion.
e. *Intertwinement*. It is important to consider an instructional sequence and its relation to other domains. Regarding the learning computational estimation,

this topic is apparently integrated in other mathematical topics: whole numbers, fractions, decimals, etc. Therefore, the following questions emerge: which are mathematical topics can support students to learn estimation? What other topics involved in learning computational estimation?

In addition to the five tenets above, there are also heuristics or principles offered by RME to design learning-teaching environments such as: *guided reinvention*, and *didactical phenomenology* (Gravemeijer, 1994).

The principle of *guided reinvention* states that students should experience the learning mathematics as a process similar to the process by which mathematics was invented under the guidance of the teacher and the instructional design (Gravemeijer, 1994, Bakker, 2004). Regarding the learning-teaching estimation, students are guided by the teacher to use estimation strategies in solving estimation problems which is supported by an instructional instrument: in our case estimation problems.

The principle of *didactical phenomenology* was developed by Freudenthal (1983), namely it concerns the relation between object and phenomenon from the perspective of teaching and learning. In particular it addresses the question how mathematical 'thought objects' can help in organizing and structuring phenomena in reality (Drijvers, 2003). In short, it refers to looking for situations that create the need to be organized (Doorman, 2005).

Thus, in the learning activity students should be allowed and encouraged to invent their own strategies and ideas in mathematical exploration and problem solving under the teacher guidance; they should learn mathematics based on their own authority in the interactive learning-teaching processes, where at the same time, their learning processes should lead to particular learning goals. Regarding the learning of computational estimation, this raises a question: how to support the students' learning processes on computational estimation to reach learning goals meaningfully?

3 Research methodology

To achieve the research aims, we support students in learning estimation with instructional activities and learning-teaching situations in the frame of the theory of realistic mathematics education. This implies we need to design an instructional environment that supports students in achieving learning goals. Because design is a crucial part of the research, we use *design research* as the research methodology.

3.1 Design research

Design research, also called design experiment or developmental research, is a type of research method in which the core is formed by classroom teaching experiments that center on the development of instructional sequences and the local instructional theories that underpin them (Gravemeijer, 2004). The purpose of this kind of research can be to develop and refine both the hypothetic of students' learning process and the means that are designed to support that learning (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). In the case of our research, the purpose is to answer the research questions about students' thinking processes and to design an instructional environment that supports students in learning estimation.

Design research encompasses three phases: developing a preliminary design, conducting a teaching experiment, and carrying out a retrospective analysis (Gravemeijer, 2004; Bakker, 2004). Before elucidating these three phases, we need to define a hypothetical learning trajectory (HLT). According to Bakker (2004), HLT is a design and research instrument that proved useful during all phases of design research. Simon (1995) defines a HLT as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (p. 136)

During the phases of the research HLT has different functions. In the preliminary design, the HLT serves as a guideline in designing instructional materials that will be used. In the teaching experiment, the HLT serves as a guideline for the teacher and researcher what to focus on in teaching, interviewing, and observing. And in the retrospective analysis, the HLT serves as a guideline in determining what the researcher should focus on in the retrospective analysis. Next, after the retrospective analysis, the HLT can be re-formulated to make a new HLT for a next design (Bakker, 2004).

In the following three subsequent sections, we describe the three phases, according to Gravemeijer (2004), Bakker (2004), and Gravemeijer and Cobb (2006), of our design research on computational estimation.

3.2 Phase 1: Preliminary design

In this phase, we formulate an HLT which consists of three components: learning goals; an instructional instrument that will be used—in our case in the form of estimation problems; and a hypothetical learning process which anticipates how students' thinking will develop. To produce the HLT we use a literature review, daily life experiences, and discussions with experienced researchers and teachers.

For this present research, in this phase, we produced what we called HLT 1 (see Chapter 4). This HLT was used in the first research period: May-June 2008. The purpose was to answer the first three research questions formulated in chapter one. This research period served: to try out the instructional activities (estimation problems), to know students' prior knowledge in estimation, and to get an initial understanding of students' thinking processes in solving estimation problems. These would also be used to revise the HLT 1. Thus, based on these functions, in this research period the students were only asked to solve estimation problems without any external intervention either from their teacher or researchers and no discussion among students. In short, students were asked to solve the problems individually. How was the research procedure of this period implemented? What data had been obtained from this period?

The procedure was as follows: (1) the researcher had prepared seven sets of estimation problems and the possible solution strategies that might be used by students; (2) each set consisted of two problems except set 1 (three problems) and was tried out to primary school students of grade four (of the second semester)—10-11 years old, where the class is a PMRI[1] class; and (3) after the trying out, the researcher selected at least three students' worksheets and interviewed the students about their thinking processes in solving the problems. Therefore, from this period we got **students' worksheets and interview data**. These data were analyzed to answer the first three research questions.

Based on the analysis of results of the first research period, we then revised HLT 1. This revision—called HLT 2—was then used for the second research period: July-August 2008 (see Chapter 5).

3.3 Phase 2: Teaching experiment

In the second phase, instructional activities are tried out during the experiment. The actual enactment of the instructional activities in the classroom enables the researchers to investigate whether the mental activities of the students correspond with the ones anticipated. The insights and experiences gained in this experiment form the basis for the design or modification of HLT for subsequent instructional activities and for new hypothesis about what mental activities of the students can be expected.

The teaching experiment took place in the second research period. Here the HLT 2 would be used. In this period, we conducted a teaching experiment for primary school students of the first semester of grade five (10-11 years old). The class that would be used, according to the teacher, consists of around 40 students. The class was different from the class of the first research period as it was a non-PMRI class.

The purposes of this research period were to get better answers to the first three research questions than in the first research period and to answer the fourth research question formulated in the previous chapter. Here, the students would be asked to solve estimation problems under the teacher guidance in learning-

[1] PMRI stands for *Pendidikan Matematika Realistik Indonesia*. It is the Indonesian version of Realistic Mathematics Education (Sembiring, Hadi, &Dolk, 2008).

teaching situations: there would be teacher explanations and groups as well as class discussions under the teacher guidance.

The teaching experiment consisted of six lessons. In each lesson—it would take 60-80 minutes—students would solve two problems. A few days later on after each lesson, at least two students would be selected for an interview based on their worksheets to know their thinking processes.

Before the teaching experiment we would have a discussion with the teacher about a plan how the teaching experiment would be implemented. The discussion would also be carried out before each lesson for 10-15 minutes. The teacher involved in this research period is the teacher of grade five. She, with 20 years experience, had been involved in the PMRI project for 3-4 years. Therefore, we expected that she has understood how to implement teaching-learning mathematics based on RME (a RME approach) which would be used in this research.

During the teaching experiment, in each lesson, every student was given a worksheet. There was an observer who would help to use a video camera. The researcher would always be available in the classroom to help the teacher during the lessons, to take pictures of important moments during learning-teaching situations, and to note important learning-teaching moments. When the teacher found problems in the lessons, she could then discuss there with the researcher. The researcher, if possible, was available to help the teacher.

How would the learning-teaching situation in each lesson take place? In each lesson, we expected that the teacher would do the following. First, the teacher introduces a topic for each lesson at the beginning of the class. She introduces by starting from experientially real activities for students that have to do with estimation problems on the student's worksheet. This was not only to make students grasp the context of problems that would be encountered easily, but also to reflect an intertwinement between mathematical topics and daily life problems. This part would take for 5-10 minutes. Second, after the introduction, students would work in groups: each student would get one worksheet. In each group, firstly each student would work individually. This was to elicit students' own strategies in solving estimation problems. Next after several minutes, 20-25 minutes, within the group, students discuss and share strategies with each other. This was to develop the same understanding in estimation as well as to reflect an interactive lesson as suggested by the RME tenets. While working in groups, the teacher would observe the students from one group to other groups. She would give guidance to students either in solving difficulties or in re-inventing estimation strategies. Third, after group discussion, the teacher would guide students to continue to class discussion. Here the teacher would then select several students from different groups to present their answers in front of the class. The students' presentation would be discussed. Therefore, we expected that there would be an interactive classroom situation: students can ask, argue, agree or disagree, etc. This learning-teaching situation was designed not only to reflect the four tenets and principles of the realistic mathematics education as mentioned in the previous chapter but also to see whether this learning situation can better support students in learning estimation.

By following a guideline from the HLT 2, we would collect data in the forms: **students' worksheets, video data, audio interview data, pictures, and field notes** during this teaching experiment. These data would be analyzed in the retrospective analysis to answer the research questions.

3.4 Phase 3: Retrospective analysis

In this phase, all data during research are analyzed so as to answer research questions. In the analysis, the HLT is compared to students' actual learning. On the basis of such analysis, we then can answer the research questions.

The result analysis of the first period, in addition to answer research questions, would also be used as a reasonable reason of the revision of the HLT 1 (see Chapter 4). And the result analysis of the second research period, in addition to answer research questions, will be used to revise the HLT 2 for future research.

3.5 Differences between PMRI and non-PMRI classes

In this section we describe PMRI and non-PMRI classes as mentioned in the sections 3.2 and 3.3 respectively because having experience with PMRI might have influence on success of the design research.

The school—from grade one to grade six—where we conducted the research has two types of classes, namely PMRI and non-PMRI classes. Each grade of this school consists of 6 classes: 1 class as a PMRI class and 5 classes as non-PMRI class. The reason of this set up is because there are not enough people to assist in the implementation of PMRI (Sembiring, Hadi, & Dolk, 2008).

In the PMRI classes the teachers should try to implement a RME approach, namely: implementing classroom learning-teaching situations by referring to the tenets and principles of RME. To do this, the teachers have been trained in implementing the RME approach (for example, attending PMRI workshops, PMRI seminars, etc); the teachers should use PMRI books which are designed by referring to the tenets and principles of RME: in this case the books were aligned with the Indonesian mathematics school curriculum (Sembiring, et al., 2008).

On the other hand, in the non-PMRI classes the teachers do not have to implement learning-teaching situations by the RME approach. Instead, they have a freedom to use whether the RME approach or not—the teachers usually use conventional teaching-learning approaches. In this type of classes the teachers use non-PMRI books which fit with the Indonesian school mathematics curriculum. Accordingly, the differences between PMRI and non-PMRI classes can be seen in the forms of the approaches and the books which are used.

In the first research period we used the PMRI class of the second semester of grade four. In this class we worked only with the students, but we did not work with the teacher, there was no teaching given to the students. They only solved estimation problems that had been designed by the researcher(s). We worked with this kind of students because we assumed that they are used to solving mathematical problems by their own strategies—because they have been taught by the RME approach since grade one.

In the second research period we used the non-PMRI class of the first semester of grade five. In this class, we would implement the lessons in the teaching experiment using the RME approach based on our HLT 2. This is done to know whether or not the RME theory that underpins the HLT 2 in this design research can support students' thinking processes in the learning estimation for the non-PMRI class's students. To do this, we worked with the teacher that has 20 years of teaching-experience and has been involved in the PMRI project for 3-4 years (she should use RME approach in the PMRI class but not in other classes). It is necessary to know that not all teachers of grade five are involved in the PMRI project. Teachers who are involved in the PMRI were selected based on: their interest in innovation of new teaching approaches (in this case the RME approach), good performances in teaching-learning, and trust by the school principal. Accordingly, we assumed the teacher—who was involved in our research—has understood how to implement the teaching-learning situations based on the RME approach and has good performances in conducting learning-teaching mathematics situations.

4 First hypothetical learning trajectory and the retrospective analysis

In the present chapter we describe the first hypothetical learning trajectory (HLT 1) which was used during the first research period and the analysis of the results of this research period. Subsequently, we first describe the didactical phenomenology which was used as a basis in designing estimation problems. Second, we describe HLT 1 which was used for primary school students of the second semester of grade four—10-11 years old. Third, we analyze the results of this research period. And fourth, we describe the revision of HLT 1 to HLT 2 which will be used in the second research period.

4.1 Brief didactical phenomenology

Bakker (2004) summarizes Freudenthal's idea about phenomenology and didactical phenomenology as follows:

> To clarify his notion of phenomenology, Freudenthal (1983) distinguished thought objects (*nooumena*) and phenomena (*phainomena*). Mathematical concepts and tools serve to organize phenomena, both from daily life and from mathematics itself. A phenomenology of a mathematical concept is an analysis of that concept in relation to the phenomena it organizes.
> …
> Didactical phenomenology: the study of concepts in relation to phenomena with a didactical interest. The challenge is to find phenomena that 'beg to be organized' by concepts that are to be taught (Freudenthal, 1983, p.32, Bakker, 2004, p.7)

In the case of our research, a world phenomenon that emerges in daily life is a situation in which we need a decision to solve calculation problems that take too much time to calculate or need calculation aids to solve such problems, for instance, supermarket problems. On the other hand, the mathematical concept that serves to aid in solving calculation (arithmetic) problems, without using calculation aids, which are most encountered in daily life, is estimation. This mathematical concept can be used to organize such a phenomenon, and it can be used to be taught in the context of educational practice: teaching-learning situations. Particularly in this research, the phenomenon is used as a basis in designing estimation problems.

4.2 First hypothetical learning trajectory

The description of the HLT 1 includes: learning goals, starting point of students' learning, intended activities, intended learning processes, and students' thinking and learning processes.

The learning goals that should be achieved by students can be classified into general and specifics goals. In general, the goal is that the students learn to use estimation strategies in solving estimation problems, while the specific learning goals are: (1) students are able to solve estimation problems with complete data, in areas of integers and rational (decimal or fractions) numbers, by estimation strategies; (2) students are able to solve estimation problems with incomplete or unavailable data, in areas of integers, by estimation strategies; (3) students would be aware with problems that they encountered in daily life whether these are estimation problems or not; and (4) students would be better estimators.

As the starting point of learning, according to the Indonesian mathematics curriculum for primary school of grade four (Depdiknas, 2006), students should already know basic arithmetic facts, i.e., addition, subtraction, multiplication, and division of integers, fractions, and decimal numbers. In the first semester of grade four the students learned rounding off numbers and also estimation, as found in the following interview.

Interviewer: *In which semester is estimation usually taught?*
Teacher: *Based on the curriculum today, we usually teach estimation for the first time in the first semester of grade four.*

Interviewer: *Is the estimation taught for a specific chapter or part of a chapter?*
Teacher: *Mmmm, usually, estimation is only part of a chapter. It is usually mixed with the topic of rounding off numbers. So, rounding off and estimation together are part of a chapter. Therefore, estimation is not taught for a whole chapter.*

Interviewer: *In which arithmetic operations is estimation taught?*
Teacher: *Mmmm, it is taught in the areas of addition, subtraction, multiplication, and division. Yes, it is taught for all basic operations in arithmetic. Although the estimation and rounding off are mixed, actually there are several differences. For rounding: rounding units to tens, tens to*

hundreds, hundreds to thousands. While estimation includes: estimate results of addition, subtraction, multiplication, and division.

For learning activities, we made 15 problems for 7 lessons: there are two versions of Problems 1 – 9 but not for Problems 10 – 15 (see Table 4.1). We made two versions of problems to elicit more different strategies in solving estimation problems, and to support students to work individually (not working together). In each lesson students would solve two estimation problems—except for lesson one which consists of three problems. In the lessons of this first period, there would be no teaching activity as explained before. Students should only solve the problems without any external intervention either from their teacher or the researcher as well as there would be no discussion among them.

Table 4.1: A summary of estimation problems used in the period: May-June 2008

Yesterday, Tom's mother went to the "Supermarket" to buy daily needs. In the Supermarket, she bought goods which appear in the receipt (see appendix 2 Figure 8.1). **Problem 1.a:** Is it enough for the mother if she uses a note of Rp 50,000 to buy all the goods? Explain your answer! **Problem 1.b:** Is it enough for the mother if she uses a note of Rp 80,000 to buy all the goods? Explain your answer!
Problem 2.a: Use the receipt in Problem 1.a. If you do not buy milk (INDOMILK), is Rp 25,000 enough to buy the goods? Explain your answer! **Problem 2.b:** Use the receipt in Problem 1.b. If you do not buy milk (INDOMILK), is Rp 20,000 enough to buy the goods? Explain your answer!
Given, the prices of two bundles of *Kangkung* are Rp 3,750 and three bundles of Spinach are Rp 4,550. **Problem 3.a:** According to you which one is cheaper, a bundle of *Kangkung* or a bundle of Spinach? Explain your answer! **Problem 3.b:** According to you which one is cheaper, 5 bundles of *Kangkung* or 5 bundles of Spinach? Explain your answer!
Given (in the picture see Figure 4.3) the prices of two bundles of *Kangkung* are Rp 3,750 and three bundles of Spinach are Rp 4,550. **Problem 4.a:** If you have Rp 10,000, is it enough to buy 5 bundles of Kangkung? Explain your answer! **Problem 4.b:** If you have Rp 15,000, is it enough to buy 7 bundles of Kangkung? Explain your answer! **Problem 5.a**: If you have Rp 15,000 how many bundles of spinach could you buy? Explain your answer! **Problem 5.b**: If you have Rp 10,000 how many bundles of spinach could you buy? Explain your answer!

Problem 6.a:	Problem 6.b:	Problem 7.a:	Problem 7.b:
28[redacted] 4[redacted] + A. 62[redacted] B. 7[redacted] C. 557	3[redacted]5 5[redacted] + A. 1[redacted] B. 96[redacted] C. 7[redacted]7	9[redacted]2 489 -- A. 4[redacted] B. 56[redacted] C. 6[redacted]	7[redacted]2 23[redacted] -- A. 4[redacted]4 B. 6[redacted]4 C. 1[redacted]

Consider the figure (see Figure 4.4). It is known that the price of 1 kg of white cabbage is Rp 1,675.
Problem 8.a: If you have Rp 10,000, is that enough to buy 5 kg of white cabbage? Explain your answer!
Problem 8.b: If you have Rp 8,000, is that enough to buy 4 kg of white cabbage? Explain your answer!

Consider the figure (see Figure 4.9). It is known that the price of 1.5 kg of chicken wings is Rp 14,000.
Problem 9.a: If you have Rp 5,000, is it enough to buy 1/2 kg of chicken wings? Explain your answer!
Problem 9.b: If you have Rp 10,000, is it enough to buy 1 kg of chicken wings? Explain your answer!

Problem 10: (Given a figure, see Figure 5.2) It is known that the price of a big ice cream is Rp 5,950 and a small ice cream is Rp 3,950.
Using Rp 20,000 how many small or big ice creams could you buy? Explain your answer!

Problem 11: Given a figure of packet A [contains 2 hats] with its price Rp 69,999, and packet B [contains three hats, see Figure 8.2 in appendix 2] with its price Rp 99,999. From the two groups of figures, which one is cheaper, the packet of A or B? Explain your answer!

Problem 12: Choose a possible right answer for a multiplication below. Then write your reasons in the provided place!

79
3[redacted] x
[redacted]
[redacted]7 +
A. 2[redacted]8[redacted]
B. 226[redacted]
C. 33[redacted]3

Problem 13: A radio sport reporter says: "This afternoon, there are 9998 supporters of PERSIB Bandung who will go to Jakarta using 19 buses, to Gelora Bung Karno stadium Jakarta, to support their team, when PERSIB Bandung against PERSIJA Jakarta…" According to you, does the news make sense? Explain your answer!

The following is a price list of fruits per kilogram in a fruit shop.

Fruits	Price/kg
Apples	Rp 11,900
Oranges	Rp 9,900
Grapes	Rp 19,900

Problem 14: If you have Rp 20,000, is your money enough to buy 1 kg of apples and $\frac{1}{2}$ kg of oranges? Explain your answer!

Problem 15: If you have Rp 25,000, is your money enough to buy $\frac{1}{2}$ kg of apples and $\frac{3}{4}$ kg of grapes? Explain your answer!

In the first and second lesson, students should solve Problems 1–5. For both versions of the problems, students were expected to increasingly solve the problems—with integers in areas of addition or subtraction, and simple multiplication—by estimation strategies.

In the third and fourth lesson, students should solve Problems 6–9. For both versions of the problems, students were expected to increasingly solve problems—with integers, decimal and simple fractions in areas of addition, subtraction, multiplication, and division—by estimation strategies.

For the lessons 5–7, students should solve problems 10–15. In this case, students were expected to increasingly solve problems—with integers, decimal and fractions in areas of addition, subtraction, multiplication, and division as well as a combination of these and also operation with decimal and fractions—by estimation strategies.

For a brief overview of the HLT 1 see Table 4.2. Detailed predictions of students' possible answers to the estimation problems 1–15 can be found in Table 8.1—a table of a comparison between HLT and students' actual strategies—in the appendix 1.

What would students' thinking processes look like during this research period? In general, **we expected that students would increasingly use estimation strategies**. This means during the lessons we predicted that there would be students who solve estimation problems by estimation strategies and there would also be other students who solve problems by exact calculation strategy. We expected number of the latter kind of students would decrease from lesson to lesson.

To stimulate students in the use of estimation strategies, we designed problems with questions that do not require exact answers; the numbers which are involved in the problems are sophisticated to make students less tempted to use exact calculation strategy, and the problems are designed to be experientially real for students. Examples of students' thinking processes in solving estimation problems will be given in the following paragraphs.

Table 4.2: An overview of HLT 1 (used in the first research period: May-June 2008)

Problems	Type of numbers	Operations	Expected Difficulty
1 – 5 (**a**/**b** versions)	Integers Examples: 50,000; 10,000; 5; etc.	Addition, subtraction, multiplication	**Easy**
6 – 9 (**a**/**b** versions)	Integers, decimals, simple fractions Examples: 1,675; 1.5; 1/2; etc.	Addition, subtraction, multiplication, division, and a combination of these	
10 – 15	Larger Integers, decimals, fractions Examples: 69,999; 9,998; 3/4; etc.	Addition, subtraction, multiplication, division, and a combination of these also operations with fractions and decimals	**Difficult**

We give two examples of predictions of what students' thinking processes would look like. For the first example, we consider the Problem 8.a above (see Table 4.1). To solve this problem, several thinking processes that would be used by students can be the following.

Since the question does not require an exact answer, first students would think for example Rp 1,675 as Rp 2,000. Then to find whether Rp 10,000 is enough or not to buy 5 kg of cabbage, one of the following calculations can be used:

- 5 x Rp 2,000 = Rp 10,000. This means that Rp 10,000 is enough to buy 5 kg of cabbage.
- Rp 10,000**:** 5 = Rp 2,000. Since Rp 2,000 > Rp 1,675, this means that Rp 10,000 is enough to buy 5 kg of cabbage.
- Rp 10,000**:** Rp 2,000 = 5. This means that 5 kg of cabbage can be bought by Rp 10,000.

Another thinking process can be the following:

- Since 1 kg of cabbage is Rp 1,675 and students are required to buy 5 kg of cabbage, this means they would do a multiplication 5 x Rp 1,675 = Rp 8,375. This is less than Rp 10,000. Therefore, Rp 10,000 is enough to buy 5 kg of cabbage.
- If students find difficulties to do a multiplication 5 x Rp 1,675, they would consider the question which does not require an exact answer. They might then try to round off the Rp 1,675 to the nearest thousand. However, since it is 'too' far, then they would round off it to the nearest hundreds, namely for example to Rp 1,600 or Rp 1,700. Then they do a multiplication 5 x Rp 1,600 = Rp 8,000. This means that Rp 10, 000 is enough to buy 5 kg cabbage.

- If students still find difficulties to do a multiplication 5 x Rp 1,600 or 5 x Rp 1,700, they would then think to round off Rp 1,675 to the nearest thousand namely Rp 2,000. Then do calculations like in the previous paragraph (like in the previous processes of thinking).

For the second example, consider the Problem 6.a above (see Table 4.1). To solve this problem, several thinking processes that would be used by students can be the following:

> Since this is an addition problem with incomplete data (an inkblot problem) they would think 28● as 280 and 4●● as 400 or 500, for example. Hence, 28● + 4●● = 280 + 400 = 680. But this answer is not included in the options. Therefore, they might then think, for example, 28● + 4●● = 280 + 500 = 780, which means the option **B** is the most possible right answer.
>
> Another thinking process can be as follow:
> Firstly, students might think 28● + 4●● as a common addition problem. Hence, they might solve it by using an addition algorithm (doing addition from the right to the left side). However, it is impossible because several numerals are covered by inkblots. Secondly, they might then try to replace the inkblots by arbitrary numerals. However, there are many possibilities. Therefore, they will see 28● and 4●● as a whole (considering positional number systems: from left to the right side). Next, they might think 28● as around 280 and 4●● as around 400. Consequently, they would do like in the previous paragraph.

For other problems, we wrote down students' possible strategies in the Table 8.1 of a comparison between HLT and students' actual strategies in the appendix 1.

4.3 Retrospective analysis: Research period May-June 2008

In this section we focus on the analysis of the use of estimation strategies of students in the first research period. As described in the previous section, we expected that students would increasingly use estimation strategies. The analysis is focused on answering the first three research questions: (1) *What strategies do students use to solve estimation problems?* (2) *What are students' difficulties in solving estimation problems?* and (3) *What kind of problems invite students to use estimation instead of using exact calculation?*

We first focused on the analysis of strategies used by students to solve estimation problems. From students' answers, as summarized in Table 4.3 (a complete table can be seen in the Table 8.1 in the appendix 1), we found—as we predicted in the HLT 1—there are two kinds of strategies used by students to solve estimation problems, i.e., estimation strategies (EST) and exact calculation strategy (EXA). Students' answers that used only words (without mathematical reasons) or no answers at all, we classified these as unclear reasons (U). Estimation strategies which were used by students can be classified as *rounding* and *front-end* strategy, where *rounding* strategy is used most. This means, in this case, the cognitive processes used by students belong to *reformulation*. Figures 4.5–4.8 are examples of students' answers using *rounding* strategy, while Figures 5.9 and 5.11 are examples of students' answers using *front-end* strategy. Why did students use only *reformulation*—instead of *translation* or *compensation*?

Table 4.3: Students' actual strategies in solving estimation problems (May-June 2008)

Problems	Students' actual strategies		
	% EST	% EXA	% U
1.a ; 1.b	11 ; 18	78 ; 76	11 ; 6
2.a ; 2.b	11 ; 18	78 ; 53	11 ; 29
3.a ; 3.b	28 ; 24	44 ; 41	28 ; 35
4.a ; 4.b	55 ; 26	36 ; 42	9 ; 32
5.a ; 5.b	50 ; 42	23 ; 16	27 ; 42
6.a ; 6.b	55 ; 58	0 ; 16	45 ; 26
7.a ; 7.b	50 ; 58	35 ; 11	15 ; 31
8.a ; 8.b	27 ; 21	64 ; 58	9 ; 21
9.a ; 9.b	18 ; 21	4 ; 0	77 ; 79
10	36	50	14
11	33	36	31
12	21	26	53
13	5	36	59
14	42	21	37
15	42	11	47

Note: EST = Estimation strategies; EXA = Exact calculation strategy; U = Unclear.

To find out reasons why students did not use other estimation strategies, we present an example of Problem 15 in Figure 4.1 below.

The following is a price list of fruits per kilogram in a fruit shop.

Fruits	Price/kg
Apples	Rp 11,900
Oranges	Rp 9,900
Grapes	Rp 19,900

Problem 15: If you have Rp 25,000, is your money enough to buy $\frac{1}{2}$ kg of apples and $\frac{3}{4}$ kg of grapes? Explain your answer!

Figure 4.1: Problem 15 (research period May-June 2008)

Using *changing operations* strategy–which belongs to *translation*, for example, Problem 15 can be solved as follows. First, we round off the prices of 1 kg of apples Rp 11,900 and 1 kg of grapes Rp 19,900 to Rp 12,000 and Rp 20,000 respectively. Hence, the solution of the problem can be the following: $\frac{1}{2}$ x 12,000 + $\frac{3}{4}$ x 20,000 = $\frac{1}{2}$ x 12,000 + (1 x 20,000 – $\frac{1}{4}$ x 20,000) = 6,000 + 20,000 – 5000 = 26,000 – 5,000 = 21,000. From this example, we can perceive that using *changing operations* strategy seems more complicated because we need to know relationships between numbers and operations, and we should be able to recreate new equivalent numbers with different operations. Because of, for example, this complicated process it might be possible that most of primary school students—in this case at the age 10-11 years old—still can not reach this cognitive process of *translation*.

Another reason can be the following. From the estimation problems themselves, we found that the numbers involved in the problems do not directly invite students to use *translation* or *compensation*. Instead, it is easier to use *reformulation*—in this case using *rounding* and *front-end* strategy. For example, instead of using *changing operations* strategy, to solve Problem 15, it is easier to use *rounding* strategy as follows: $\frac{1}{2} \times 12{,}000 + \frac{3}{4} \times 20{,}000 = \frac{1 \times 12{,}000}{2} + \frac{3 \times 20{,}000}{4} = \frac{12{,}000}{2} + \frac{60{,}000}{4} = 6{,}000 + 15{,}000 = 21{,}000$.

Accordingly, to solve estimation problems most of the students would use estimation strategies that they find easy—namely *rounding* strategy. In case of the use of *front-end* strategy, it could be because students understand the positional system of numbers. For example, we can see in Figure 5.9, to solve 28● + 4●● using *front-end* strategy, students solved as follows. First 200 + 400 = 600, but it is written 28●. which can mean 280 + 400 = 680. Hence, if we added up 280 and 400, then the sum is more than 600.

Based on the description above we can say that estimation strategies which were used by primary school students (of grade four) to solve estimation problems include only *rounding* and *front-end* strategy.

As a second step in the analysis we present the graph in Figure 4.2—of overall percentages of students using estimation—as an overview of students' global performances in the use of estimation during the first research period.

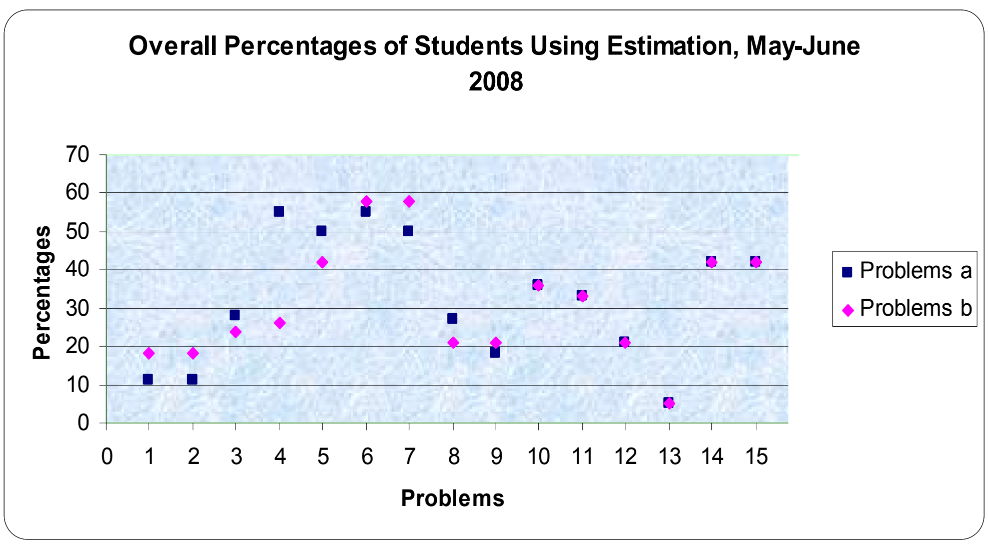


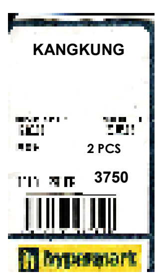
Figure 4.2: Overall percentages of students using estimation in the period May-June2008

Note: Problems 1 to 9 have **a** and **b** versions, whereas Problems 10 to 15 do not have versions.

From the graph in Figure 4.2 we see that for problems 1 to 7 there is an upward trend in the use of estimation strategies, both for the **a** and **b** versions. This trend, however, does not continue. From the graph we can also make the following observations: (1) there is only a small difference in the use of estimation between the **a** and **b** versions of problems 1 to 9, except for 4; and (2) there is a sudden drop in the use of estimation strategies after Problem 7. This was a contradiction to our expectation in HLT 1. We therefore go on to further analyze the data in search of a possible explanation.

Observation 4.1: Difference in the use of estimation of Problems 4.a and 4.b

There is a large difference between Problem 4.**a** and Problem 4**.b** in the use of estimation. To find out reasons for this difference, we compare the two versions of the Problems (see Figure 4.3).



Problem 4.a: If you have Rp 10,000, is it enough for you to buy 5 bundles of *Kangkung*? Explain your answer!

Problem 4.b: If you have Rp 15,000, is it enough for you to buy 7 bundles of *Kangkung*? Explain your answer!

Note: For both versions of the problem, the next figure is given.

Figure 4.3: Problems 4.a and 4.b (research period May-June 2008)

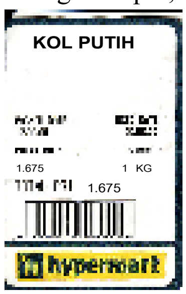
In Problem 4.a, we use smaller numbers than in problem 4.b**:** 10,000 < 15,000 and 5 < 7; and multiplication or division by 5 is easier than by 7; besides that 5 is a factor of 10, whereas 7 is not a factor of 15, so calculation with 5 is easier.

Based on the analysis above, we concluded that Problem 4.a is easier to solve than Problem 4.b, and hence inviting more students to use estimation strategies. Therefore, Problem 4.a was used again in the second research period, but problem 4.b was not used anymore.

Observation 4.2: A sudden drop in the use of estimation after Problem 7

We discuss Problems 8, 9, 12 and 13 because these problems have lower percentages than other problems in the use of estimation strategies. To find reasons for this observation, we first discuss Problem 8 as shown in Figure 4.4 below.

Consider the figure below. From the figure, it is known that the price of 1 kg of white cabbage is Rp 1,675.



Problem 8.a: If you have Rp 10,000, is that enough to buy 5 kg of white cabbage? Explain your answer!

Problem 8.b: If you have Rp 8,000, is that enough to buy 4 kg of white cabbage? Explain your answer!

Figure 4.4: Problems 8.a and 8.b (research period May-June 2008)

Because Problem 8 was given in the fourth lesson, we expected that students had a sufficient experience in the use of estimation strategies. Therefore, for Problem 8.a, in the HLT 1 we expected that students would use one of the possible estimation strategies below.

- Since the question only requires to knowing whether Rp 10,000 is enough or not to buy 5 kg of white cabbage, and it is known that each kg costs Rp 1,675, then we expected that students would estimate the price of 5 kg of white cabbage by rounding off the price/kg Rp 1,675 to Rp 2,000 for example. Therefore, students would easily find that Rp 10,000 is enough to buy 5 kg of cabbage because Rp 2000 x 5 = Rp 10,000,-
- Students might estimate the price of 5 kg of white cabbage by rounding off Rp 1,675 to other easy numbers, for example to Rp 1,800. Next, students would do a multiplication Rp 1,800 x 5 = Rp 9,000 which is less than Rp 10,000. Hence, they would conclude that Rp 10,000 is enough to buy 5 kg of white cabbage.
- Students might see the problem as a division problem, namely Rp 10,000**:** 5 = Rp 2,000. This is then compared to Rp 1,675. Hence, this means Rp 10,000 is enough to buy 5 kg of cabbage.

However, students who still do not see the problem as an estimation problem might solve the problem using an exact calculation strategy, for example as follows. Since 1 kg of white cabbage is Rp 1,675, then 5 kg are Rp 1,675 x 5 = Rp 8,375, and this is less than Rp 10,000. Therefore, they would conclude that Rp 10,000 is enough to buy 5 kg of white cabbage.

Using a similar reason to Problem 8.a, to solve Problem 8.b in the HLT we also expected that students would use one of the possible estimation strategies we discussed above, but with different numbers.

From the possible solution strategies above, we see that both problems (8.a and 8.b) might be easy for students of the second semester of grade four because each problem only needs a direct multiplication or division. As a consequence, most of the students tend to solve it using exact calculation strategy, which is more convincing for them than estimation. Another reason can be as follows: since rounding off Rp 1,675 to Rp 2,000 is "too far", as a consequence, most of the students could not see that estimation strategies (for example 5 x Rp 2,000 and 4 x Rp 2,000) are easier than the exact calculation strategy. Instead, they would round off Rp 1,675 to closer numbers, for example to Rp 1,600, Rp 1,700, or even Rp 1,500. Therefore, rather than doing a multiplication Rp 1,700 x 5, for example, students might tend to do an exact multiplication Rp 1,675 x 5 because the latter is more convincing for them to get an exact answer and it is not really more difficult than doing the first multiplication.

We found an interesting result in the students' answers of Problem 8: there is no student who solved the problem, for example Problem 8.a, using a division Rp 10,000 **:** 5 = Rp 2,000 and compare it with the price Rp 1,675 to conclude that the money is enough to buy 5 kg of white cabbage. Why did they not use that strategy? Possible explanations can be the following. First, most of the students would focus on the price of cabbage Rp 1,675 and the number of cabbage that should be bought, namely 5 kg. Therefore, they would use multiplication between the number of kg of cabbage and the price per kg of the cabbage. Second, it might be possible that the problem is not realistic for students because having Rp 10,000 to buy goods in a supermarket is uncommon. Hence, they would only focus on finding an answer to the problem by concentrating on the price per kg of cabbage and the number of kg of the cabbage that they should buy.

Although both versions of the Problem 8 tended to be solved using exact calculation strategy, this problem was inviting students to use more various estimation strategies—see Figures 4.5-4.8. This makes Problem 8

potentially suitable to invite students in the use of different estimation strategies. Therefore we used this problem again in the second research period.

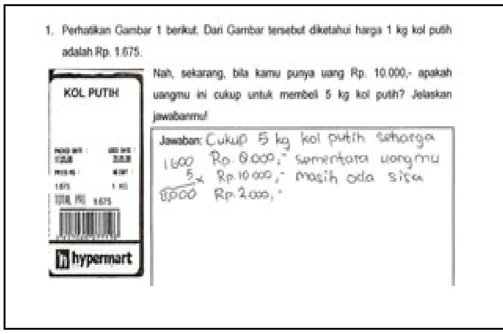


Figure 4.5: Putri's answer to Problem 8.a
Translation: Enough, 5 kg of white cabbage are Rp 8,000 [1,600 x 5], while your money is Rp 10,000; So remaining money is Rp 2,000

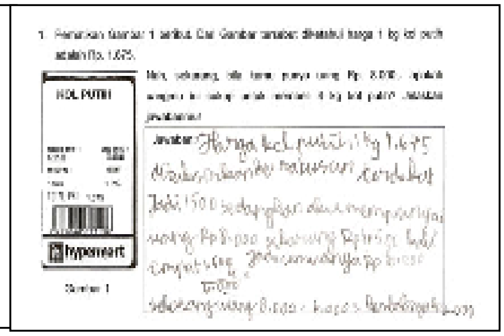


Figure 4.6: Fakhri's answer to Problem 8.b
Translation: The price of 1 kg is [Rp] 1,675, it is rounded off to [Rp] 1,500. I have Rp 8,000. Now, Rp 1,500 is multiplied by 4, to get Rp 6,000. So, the remaining money is 8,000 – 6,000 namely Rp 2,000

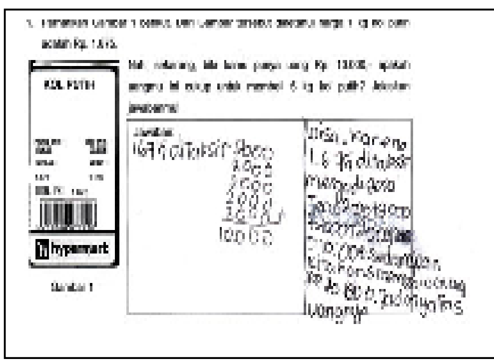


Figure 4.7: Mutya's answer to Problem 8.a
Translation: [Enough] because 1,675 is rounded off to 2000; 2000 + 2000 + 2000 + 2000 + 2000 = 10,000. We have Rp 10,000 which means our money is fit [enough]

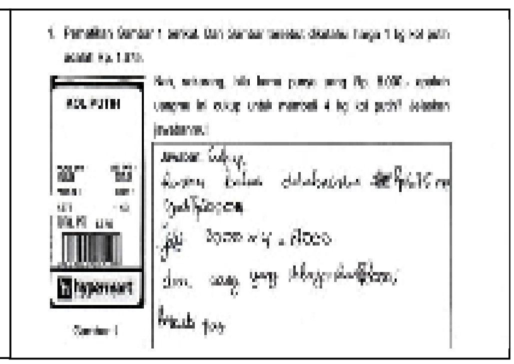


Figure 4.8: Arini's answer to Problem 8.b
Translation: Enough, because if Rp 1,675 is rounded off to Rp 2,000; then 2,000 x 4 = 8,000. And we should pay Rp 8,000. This means fit.

Since both Problems 8 and 9, **a** and **b** versions, were given in the same lesson, we also expected that students had sufficient experience to solve Problem 9 by estimation strategies. However, the results tell us differently: only 18% and 21% of the students used estimation strategies to solve the Problems 9.a and 9.b respectively. For the analysis, we show Problem 9 in Figure 4.9 and its possible solution strategies below.

Consider the figure below. From the figure it is known that the price of 1.5 kg of chicken wings is Rp 14,000.

Figure 2

Problem 9.a: If you have Rp 5,000, is it enough to buy 1/2 kg of chicken wings? Explain your answer!

Problem 9.b: If you have Rp 10,000, is it enough to buy 1 kg of chicken wings? Explain your answer!

Figure 4.9: Problems 9.a and 9.b (research period May-June 2008)

In the HLT, we expected students to use one of the following possible estimation strategies:

Because the price of 1.5 kg of chicken is Rp 14,000, then to find the price of 1/2 kg of it can be done by solving a division Rp 14,000**:** 3. However, because this is difficult, we expected students would use one of the possible estimation strategies below.

- It is known that 1.5 kg of chicken is Rp 14,000, it will be easy if the price is rounded off to Rp 15,000. Hence, the price of 1/2 kg of chicken is 1/3 of Rp 15,000 = Rp 5,000. However, this is more than the real price of the 1/2 kg of chicken. Accordingly, Rp 5,000 is enough to buy 1/2 kg of chicken.
- It is known that 1.5 kg of chicken is Rp 14,000, therefore, 3 kg of chicken are Rp 28,000 which is less than Rp 30,000. Consequently, 1 kg of chicken is less than Rp 10,000, which means 1/2 kg of chicken is less than Rp 5,000.
- If 1 kg of chicken is Rp 10,000, then 1/2 kg of it is Rp 5,000. Consequently, 1.5 kg of chicken is Rp 15,000. However, the real price of 1.5 kg is Rp 14,000 < Rp 15,000. Therefore, it is enough to use Rp 5,000 to buy 1/2 kg of chicken.
- If 1/2 kg of chicken is Rp 5,000, then 1.5 kg of it is Rp 15,000 > Rp 14,000. Therefore, it is enough to buy 1/2 kg of chicken using Rp 5,000.
- Since 1.5 kg of chicken is Rp 14,000, then 1/2 of 1.5 kg, or 3/4 kg, of chicken is Rp 7,000. It is easy to think the price of 3/4 kg of chicken as Rp 7,500; therefore 1/4 kg of chicken is Rp 2,500. This means that 1/2 kg of it is Rp 5,000 (but this is more than the real price).

For students who still did not see this problem as an estimation problem, they might solve the problem as follows. First, although it is difficult, they would find the price of 1/2 kg of chicken by dividing Rp 14,000 by 3, namely Rp 14,000**:** 3 = Rp 4,666.7. Hence, they would conclude that Rp 5,000 is enough to buy 1/2 kg of chicken.

> In a similar reason, we expected that students would be more invited to solve Problem 9.b using the possible estimation strategies similar to the ones above, but with different numbers.

From the problems and their possible solution strategies above, we in retrospect understand that these problems are relatively difficult. This is because to solve the problems we first need to translate the problems and we use extra steps more than only multiplication and division. Moreover, the problems themselves include decimal numbers, fractions, knowledge of mass unit (kg), and complex operations (a combination of division and multiplication or the other way round). These difficulties can be found from students' answers below.

- Several students did not understand the problem, especially decimals. See an example in Figure 4.10.

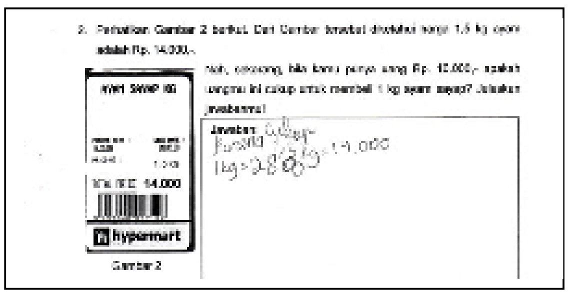


Figure 4.10: Fasya's answer to Problem 9.b.
Translation: It is enough because 1.5 kg of chicken = [Rp] 14,000. [So] 1 kg = 28,000.

- Several students did not understand decimals and fractions. See Figure 4.11 as an example.

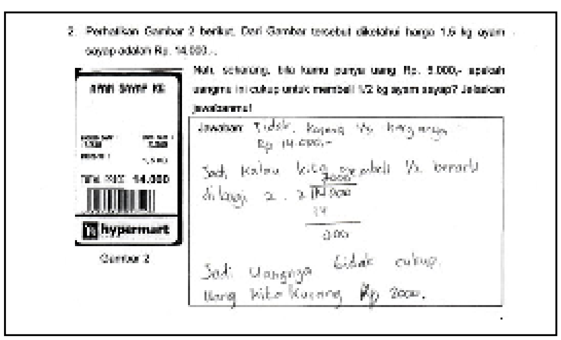


Figure 4.11: Adelia's answer to Problem 9.a
Translation: No, it is not enough because 1/5 [kg of chicken] is Rp 14,000. Therefore, if we need 1/2 [kg] it means 14,000: 2 = 7,000. Therefore, the money is not enough, we need Rp 2,000 more.

- Only two students arrived at fully right answers: one using exact calculation strategy another using estimation. See Figures 4.12 and 4.13 below.

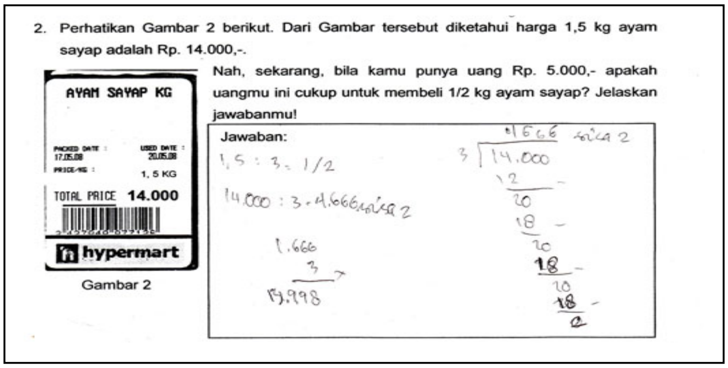


Figure 4.12: Ghifara's answer to Problem 9.a**,** using exact calculation

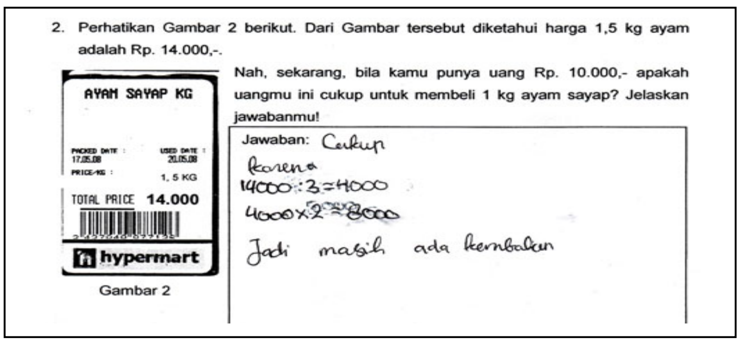


Figure 4.13: Arini's answer to Problem 9.b, using estimation
Translation**:** Enough because 14,000**:** 3 = 4,000; 4,000 x 2 = 8,000. So, there is still remaining.

Because we thought these two problems, 9.a and 9.b, were too difficult for students, the problems were not used anymore in the revision of the HLT.

We go on to analyze Problems 12 and 13. First, we show Problem 12, in Figure 4.14, and its possible solution strategies.

Problem 12: Choose a possible right answer for a multiplication below. Then write your reasons in the provided place!

```
   79
   3●  x
  ●●●
 ●●7   +
A. 2●8●
B. 226●
C. 33●3
```

Figure 4.14: Problem 12 (research period May-June 2008)

In the HLT, we expected that students would see the problem as an estimation problem, where it can be estimated as a multiplication 80 x 30 or 80 x 40. So, the multiplication 79 x 3● has results between around 80 x 30 = 2400 and around 80 x 40 = 3200. Consequently, options B and C are impossible. Therefore, the most possible right answer is option A.

However, students might solve the problem using an exact calculation strategy. To solve a multiplication 79 x 3●, students would make trial and error to replace the inkblots, next they do an exact multiplication. For example, they might do a multiplication 79 x 30, 79 x 31, 79 x 32, … or 79 x 39. Then, they might choose a possible right answer from the options based on their calculation.

Although the possible estimation solution strategies above initially seemed simple for us, most of the students, however, did not solve the problem using estimation strategies. The reasons for this can be the following. First, according to the teacher, students, and also the school mathematics curriculum, students had never encountered such a problem previously in their school career. Therefore, they might not recognize the problem as an estimation problem. Consequently, students would solve the problem using an exact calculation strategy. The following, Figure 4.15, is an example of the student's answer using an exact (trial and error) calculation strategy.

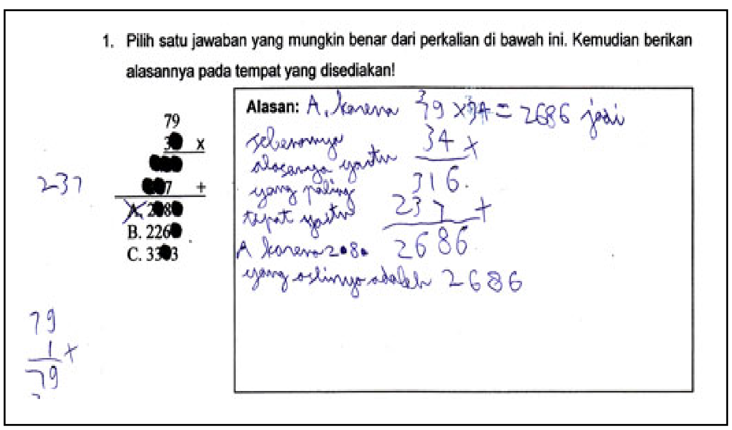


Figure 4.15: Faris' answer to Problem 12, using an exact (trial and error) calculation strategy
Translation: Actually, the true answer is A because 2●8● is 2686[79 x 34]

Second, it might be possible that students find it easier to use an exact (trial and error) strategy, for example by replacing the inkblots by zeros.

Third, because numbers involved in the problem are covered (by ink), students should decide by themselves which numbers will be used. To do this, they should know limits, both upper and bottom limit for each number. Therefore, there are two processes: students might think to multiply the two numbers using bottom and upper limits. No wonder this kind of problems is relatively difficult for most of the students.

The fourth possible reason is: it could be because most of the students are used to doing multiplications by an algorithm. Consequently, when solving the (column) multiplication (like Problem 12), students would work from the right to the left, not the other way round. In other words, students are still not aware of positional system of numbers, namely understanding the magnitude of numbers—which can be understood if they look from the left to the right. Therefore, students tend to use algorithm rather than understand the magnitude of numbers (which is important for doing estimation).

After all, there are several students' answers of this problem using estimation strategies. For example, see Figure 4.16.

Based on the analysis above, although Problem 12 is difficult for most of the students, we thought this problem is important in inviting students to understand the magnitude of numbers (which is important to do rounding off numbers that can be used for doing estimation). Therefore, we used this problem in the second research period as well.

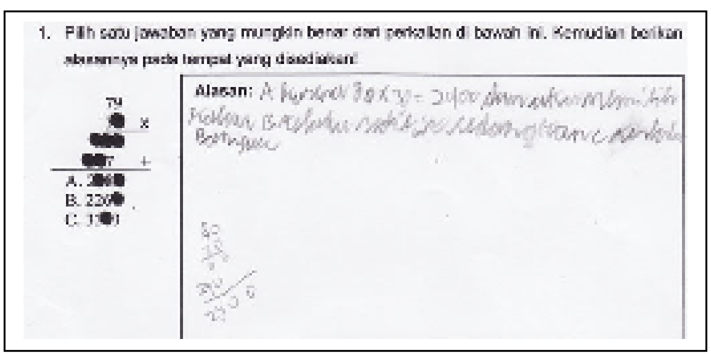


Figure 4.16: Jodi's answer to Problem 12, using an estimation strategy
Translation: [The answer is] A because 80 x 30 = 2400. If I choose B it is less, if I choose C it is greater.

Now, we analyze Problem 13, which is a kind of estimation problems with unavailable data. First we show it, in Figure 4.17, and its possible solution strategies.

A radio sport reporter says: "This afternoon, there are 9998 supporters of PERSIB Bandung who will go to Jakarta using 19 buses, to Gelora Bung Karno stadium Jakarta, to support their team, when PERSIB Bandung against PERSIJA Jakarta…" According to you, does the news make sense? Explain your answer!

Figure 4.17: Problem 13 (research period May-June 2008)

In the HLT, we expected that students would round off the numbers to 10,000 and 20—because 9998 and 19 are close to 10000 and 20 respectively. Next, we expected students to use one of possible estimation strategies below.

- Students would then find by a division that the information means each bus should be able to bring around 500 passengers (supporters). However, we expected that students would use their knowledge or experience that in a bus the maximum passengers are between 40 and 60 persons. Therefore, they finally would conclude that the news does not make sense.
- From their experience or knowledge, students were expected to use the maximum number of passengers in a bus, namely between 40 and 60, to do a multiplication 20 x 60 = 1200, for example, and compare this result to the number of passengers that should be brought by the available buses, namely around 10000. Therefore, they would conclude that the news does not make sense because 1200 is too far from 10000.

Students who did not see this problem as an estimation problem might solve by an exact division 9998:19 ≈ 526 or 527; or an exact multiplication 19 x 60 for example. Hence they would conclude that the news does not make sense. However, students who did not realize about the maximum number of passengers in a bus, only doing computation for example, might conclude that the news makes sense.

Based on the possible estimation solution strategies above, reasons why this problem is less in the use of estimation, for most students, can be the following. First, to solve this kind of problems students should invent realistic data by themselves; they must apply their knowledge of measures, for example, to solve Problem 13 students should know how many seats are likely to be in a bus. Second, students should be able to think reflectively: they should be aware not only to calculate numbers in the problem but also to decide whether what they did was reasonable or not. Third, students had never encountered this kind of problems according to the teacher and the curriculum. Therefore, it could be possible that students would not see this problem as an estimation problem. Consequently, students would solve the problem using an exact calculation strategy. And fourth, it could be because students are used to solve problems with all information given, and solve problems with fixed procedures, not combining procedures with real-world information. The following, in Figure 4.18, is an example of the students' answer.

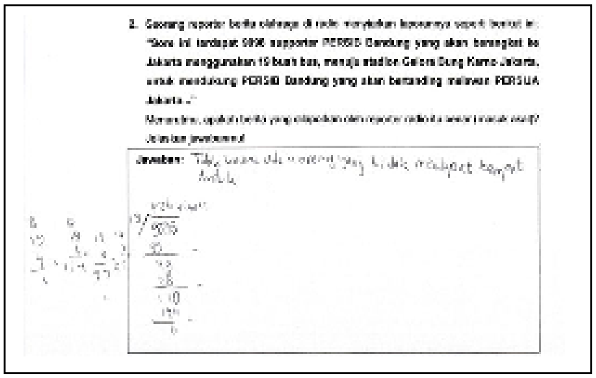


Figure 4.18: Audy's answer to Problem 13, using an exact calculation strategy
Translation: It does not! Because there are 4 people who do not get a place (seats).

Although this problem is difficult for most students in the use of estimation, we thought this problem is very important to invite students to think reflectively and reasonably, combine given and real-world information in solving the problem. Therefore, we used this problem again for the second research period.

4.4 Revision of the first hypothetical learning trajectory

In the first research period, we used 15 problems: Problems 1-9 have **a** and **b** versions, whereas Problems 10-15 do not have versions. In the second period we only used 12 problems (which came from the first period), in particular, based on the analysis in section 4.3. We decided to use the **a** version in the second period because, based on the analysis, there is only a small difference in many aspects—difficulties, type of problems, kind of numbers and operations—between the **a** and **b** versions. In the following we specify the reasons why and which problems were used or not in the second research period.

(1) **Problem 1.a** (see Table 4.1): We thought that this problem is very important as an introduction in the teaching experiment to invite students to use estimation because it uses a contextual situation (receipt from a supermarket) which is experientially real for students. Moreover, this problem is relatively easy to solve since all information is given; the problem needs only addition to arrive at a right answer; and the numbers involved in the problem are only integers. Therefore, we used this problem again in the second research period as Problem 1.

(2) **Problem 2.a** (see Table 4.1)**:** We thought that this problem is similar to Problem 1.a. The difference is at least at the use of subtraction to solve the problem. As a consequence, we decided not to use this problem anymore.

(3) **Problem 3.a** (see Table 4.1)**:** Mathematically seen, to solve this problem we need to apply multiplication or division. Because of these operations, the numbers involved could be decimals, not only integers. Hence, we thought this problem is relatively difficult, but it is necessary to invite students in the use of estimation as a way of problem solving. Therefore, we used this problem again as Problem 3 in the second research period.

(4) **Problems 4.a and 5.a** (see Table 4.1)**:** These two problems are similar. However, Problem 5.a is more difficult than 4.a, therefore this problem was removed. A complete reason why Problem 4.a was used again in the second period as Problem 4 can be found in section 4.3.

(5) **Problems 6.a and 7.a** (see Table 4.1)**:** Although these two problems are addition and subtraction problems, however, to solve these students should produce by themselves incomplete information because the numbers are covered by inkblots—this is a kind of incomplete data problems. We thought these two problems would invite students to use estimation. Therefore, in the second period, these two problems were used again as Problems 9 and 10 respectively.

(6) **Problems 8.a and 9.a** (see Table 4.1)**:** In the second period, Problem 8.a became Problem 2 and Problem 9.a was removed. The reasons for this can be found in the section 4.3.

(7) **Problem 10** (see Table 4.1)**:** We thought this problem is good because it uses a contextual situation which is experientially real for students, i.e., an ice cream context. In addition, as we can see from the graph in Figure 4.2, around 35 % of students solved this problem by estimation strategies. Therefore, we thought this problem was potential in the use of estimation. As a consequence, this problem was used again as Problem 5 in the second research period.

(8) **Problem 11** (see Table 4.1)**:** In Figure 4.2, we see this problem was potential in the use of estimation: around 35% students solved this problem by estimation strategies. In addition, we thought the numbers involved in the problem would guide students to use rounding off numbers, for example from Rp 69,999 to Rp 70,000. Operations that would be used to solve this problem are division or multiplication; types of numbers involved in this problem can be integers and decimal numbers. As a consequence, we expected that students would use estimation strategies to solve it. Therefore, this problem was used again in the second research period as Problem 6.

(9) **Problems 12 and 13** (see Table 4.1): Problems 12 and 13 became Problems 11 and 12 respectively in the second research period. The complete reasons for this can be found in the section 4.3.

(10) **Problems 14 and 15** (see Table 4.1)**:** We thought these two problems are relatively difficult for most of students because the types of numbers involved are not only integers but also rational (fractions) numbers. We would agree that operations with fractions are difficult for most of students. However, since these problems intertwine between mathematical topics (integers and fractions) and potential in the use of estimation: around 40% students solved these problems by estimation strategies, we decided to use these problems again in the second research period as Problems 7 and 8 respectively.

As a summary, we changed the order of problems for the second research period because of the following considerations: (1) on the apparent difficulty of the problems: we rearranged the order of Problems 1-12 from simple to difficult;

(2) on the kind of problems: problems with complete data were given first (in this case Problems 1-8), problems with incomplete or unavailable data were given afterward (in this case Problem 9-12) because based on the analysis in section 4.3, in general the latter kind of problems is relatively more difficult than the former; (3) on the use of operations: problems with addition or subtraction were given first, problems with multiplication or division were given later on; and (4) on the kind of numbers: problems which use only integer numbers were given first and problems with rational numbers or so were given afterward. Based on these considerations, we then rearranged the order of the problems as we can see in Table 4.4. This means the HLT 1 was revised to the HLT 2 (see Chapter 5).

Table 4.4: Order of problems used in the second research period: July-August 2008

P. May-June	1.a	8.a	3.a	4.a	10	11	14	15	6.a	7.a	12	13
P. July-August	1	2	3	4	5	6	7	8	9	10	11	12

Note: This table means, for example, Problem 1 (in the second research period) = Problem 1.a (in the first research period); Problem 2 = Problem 8.a; and so forth.

5 Second hypothetical learning trajectory and the retrospective analysis

In a similar manner to Chapter 4, we here describe HLT 2 that was used in the second research period, analyze the results of the second research period, and give a proposal to revise the HLT 2 based on the analysis. The description of the HLT 2 is similar to the HLT 1. Next, in the analysis we discuss students' actual strategies in solving estimation problems and compare these with our predictions in the HLT 2 (see Table 8.2 in the appendix 1). Our analysis, in particular, is focused on answering the research questions: (1) *What strategies do students use to solve estimation problems?* (2) *What are students' difficulties in solving estimation problems?* (3) *What kind of problems invite students to use estimation?* and (4) *What kind of learning-teaching situations invite students to use estimation?*

5.1 Second hypothetical learning trajectory

Based on the analysis of results of the first research period, as described in Chapter 4, we revised the HLT 1 to HLT 2. The HLT 2 has similarities with the HLT 1 on the learning goals and the starting points of students' learning. The differences between these are: the order as well as the number of problems and on the students' thinking processes. We can see the similarities and the differences between the HLT 1 and the HLT 2 by comparing Table 4.2 (in Chapter 4) and Table 5.1.

We repeat the same question as in section 4.2 (see Chapter 4): what would students' thinking processes look like during the second research period? In general, like in the HLT 1, **we expected that students would increasingly use estimation strategies from lesson to lesson**. This means during the lessons we predicted that there would be students who solve estimation problems by estimation strategies and there would also be other students who solve problems by an exact calculation strategy. We expected that number of the latter kind of students would decrease from lesson to lesson except perhaps for new type of problems. We predicted this would happen because: like in the HLT 1 we had designed problems with: the questions do not require exact answers and the numbers which are involved in the problems are sophisticated (to make students less tempted to use an exact calculation strategy). Moreover, in the lessons, there would be groups as well as class discussion under the teacher guidance, so students would share and learn from each other about estimation strategies. We describe our prediction of students' thinking processes from lesson to lesson in the following paragraphs.

Table 5.1: An overview of HLT 2 (used in the second period: July-August 2008)

Problems	**Type of problems**	**Type of numbers**	**Operations**	**Expected difficulty**
1 – 4	Problems with complete data	Integers Examples: 50,000; 1,675, etc.	Addition, Multiplication, Combination: division and multiplication	**Easy**
5 – 8	Problems with complete data	Larger integers, decimals and fractions Examples: 69,999; 3/4; etc.	Combination: addition, multiplication, division of integers; Combination: addition, multiplication with a (simple) fraction; Combination: addition and multiplication with fractions	
9– 12	Problems with incomplete or unavailable data	Integers Examples: 9998; 28●; 4●●, etc.	Addition, subtraction, multiplication, Combination: multiplication, division	**Difficult**

In the lessons 1 and 2, students should solve Problems 1 – 4 (see Tables 5.1 and 5.2). Since the students are in the first semester of grade five (10–11 years old), we predicted that they would be able to solve integer estimation problems in areas of addition, subtraction, (simple) multiplication or divisions.

Because mathematics is viewed mostly as a knowledge with exact answers (Trafton, 1986) and students are also used to finding exact calculation results when learning mathematics at school (Van den Heuvel-Panhuizen, 2001), then when at the first time students were given estimation problems, we predicted that they would solve estimation problems using one or a combination of the following strategies:

(1) *Exact calculation strategy*: we predicted most of students would solve estimation problems by an exact calculation strategy even if the problems only require estimate answers. There might also be students who give estimate, but the processes are first they would use exact calculation strategy to find answers, then finally estimate the final answers.

(2) *Estimation strategies*: there might also be students that see the problems require only estimate answers, so they would be invited to solve the problems by estimation strategies.

Table 5.2: A summary of estimation problems of the second period: July-August 2008

<table>
<tr><td>Yesterday, Tom’s mother went to the “Supermarket” to buy daily needs. In the Supermarket, she bought goods which appear in the receipt (see Figure 8.1 in appendix 2).

Problem 1: Is it enough for the mother if she uses a note of Rp 50,000 to buy all the goods? Explain your answer!</td></tr>
<tr><td>Consider the figure (see Figure 4.4). It is known that the price of 1 kg of white cabbage is Rp 1,675.

Problem 2: If you have Rp 10,000, is that enough to buy 5 kg of white cabbage? Explain your answer!</td></tr>
<tr><td>Problem 3: Given, the prices of two bundles of Kangkung are Rp 3,750 and three bundles of Spinach are Rp 4,550. According to you which one is cheaper, a bundle of Kangkung or a bundle of Spinach? Explain your answer!</td></tr>
<tr><td>Problem 4: Given, the price of two bundles of Kangkung are Rp 3,750 and three bundles of Spinach are Rp 4,550. If you have Rp 10,000, is it enough to buy 5 bundles of Kangkung? Explain your answer!</td></tr>
<tr><td>Problem 5: Given a figure (see Figure 5.2) It is known that the price of a big ice cream is Rp 5,950 and a small ice cream is Rp 3,950. Using Rp 20,000; how many small or big ice creams could you buy? Explain your answer!</td></tr>
<tr><td>Problem 6: Given a figure of packet A [contains 2 hats] with its price Rp 69,999, and packet B [contains three hats, (see Figure 8.2 in appendix 2)] with its price Rp 99,999. From the two groups, which one is cheaper, the packet of A or B? Explain your answer!</td></tr>
<tr><td>The following is a price list of fruits per kilogram in a fruit shop.

<table>
<tr><td>Fruits</td><td>Price/kg</td></tr>
<tr><td>Apples</td><td>Rp 11,900</td></tr>
<tr><td>Oranges</td><td>Rp 9,900</td></tr>
<tr><td>Grapes</td><td>Rp 19,900</td></tr>
</table>

Problem 7: If you have Rp 20,000, is your money enough to buy 1 kg of apples and 1/2 kg of oranges? Explain your answer!
Problem 8: If you have Rp 25,000, is your money enough to buy 1/2 kg of apples and 3/4 kg of grapes? Explain your answer!</td></tr>
</table>

Problem 9:	**Problem 10:**	**Problem 11:**
28[redacted] 4[redacted] + A. 62[redacted] B. 7[redacted] C. 557	9[redacted]2 489 -- A. 4[redacted] B. 56[redacted] C. 6[redacted]	79 3[redacted] x [redacted] [redacted]7 + A. 2[redacted]8[redacted] B. 226[redacted] C. 33[redacted]3
Problem 12: A radio sport reporter says: "This afternoon, there are 9998 supporters of PERSIB Bandung who will go to Jakarta using 19 buses, to Gelora Bung Karno stadium Jakarta, to support their team, when PERSIB Bandung against PERSIJA Jakarta…" According to you, does the news make sense? Explain your answer!		

The following are our expectation to students in solving integer estimation problems 1-4 (see Tables 5.1 and 5.2):

- For Problem 1, students were expected to solve the problem by estimation strategies in the area of addition. To make students less tempted to use an exact calculation strategy, the question uses enough or not question, and numbers involved in the problem are difficult to be added up by an exact calculation strategy.
- For Problem 2, students were expected to solve the problem by estimation strategies in the area of simple multiplication—which means students only need to do direct multiplication. Since the problem uses enough or not question and difficult numbers are same as the Problem 1, we expected students to be less tempted to use an exact calculation strategy to find an answer.
- For Problem 3, students were expected to solve the problem by a simple division, multiplication, or a combination of these. To stimulate students to use estimation strategies: numbers involved in the Problem 3 are designed not easy to divide or multiply exactly, but they are easier to round off before doing a division or multiplication; and the question used is asking a comparison without a need to know an exact answer.
- For Problem 4, students were also expected to solve the problem in areas of multiplication and division, or a combination of these by estimation strategies. To make students less tempted to use an exact calculation strategy, numbers

involved in the problem are difficult to be calculated; and the question used is asking enough or not without a need to know an exact answer.

In the lessons 3 and 4, students should solve Problems 5 – 8. Here, we expected that students had built on to use previous experiences (in the lessons 1 and 2) in solving estimation problems by estimation strategies. Therefore, they were expected to be able to solve estimation problems in areas of a combination of multiplication, division, and fractions (multiplications which involve fractions) by estimation strategies.

Through groups and class discussions, we expected that students would be more aware of effectiveness of the use of estimation strategies instead of using an exact calculation strategy. In addition, we also expected that students would use: more various estimation strategies—not only *rounding* or *front-end* strategies, but also other strategies that belong to *translation,* or *compensation*, and more effective estimation strategies than before. Furthermore, students were expected to use either estimation strategies or an exact calculation strategy flexibly depend on what problems they encounter.

The following are our expectation to students in solving estimation problems 5 – 8 (see Tables 5.1 and 5.2):

- For Problem 5, students were expected to solve estimation problem in areas of multiplication and addition or a combination of these. To make students less tempted to use an exact calculation strategy, numbers involved in the problem are made difficult but easier to be rounded off; and the question used is asking enough or not. To make students elicit more various strategies and different answers, the problem is made open with different answers.
- For Problem 6, students were expected to solve the problem by estimation strategies in areas of a combination of addition, multiplication, or division. Here students should use numbers which are easier to be calculated by rounding off to stimulate the use of estimation strategies. Besides that, the question used is asking a comparison without a need an exact calculation.

- For Problem 7, students were expected to solve the problem by estimation strategies in areas of a combination of a simple fraction (or division), multiplication, and addition. Here numbers involved in the problem are relatively complicated but easy to be rounded off. So, solving by estimation strategies would be easier than using an exact calculation strategy. In addition, the question used in this problem does not demand an exact answer because it uses enough or not question.
- For Problem 8, students were expected to solve the problem in areas of a combination of (simple) fractions, division, multiplication, and addition. Using similar reasons to Problem 7, we expected that students would solve this problem by estimation strategies.

Problems 1 – 8 are problems with complete data (problems with all data, numbers for example, are stated clearly in the problems). Having experience in solving this type of estimation problems, we expected that students would be able to solve estimation problems with incomplete or unavailable data in the next lessons.

In the lessons 5 and 6, students would solve Problems 9 – 12. Here, students were expected to recognize estimation problems with incomplete or unavailable data in areas of addition, subtraction, multiplication, or combination of these. So, they would use estimation strategies to solve the problems. However, since this type of problems are new—students had never encountered such problems before—it might be possible that students would not recognize the problems as estimation problems. Therefore, students might solve the problems by an exact calculation strategy.

The following are our expectation to students in solving estimation problems 9 –12 (see Tables 5.1 and 5.2):

- For Problems 9 and 10, because students had never encountered such problems before, we predicted that they might solve these by an exact calculation strategy—for students who did not see the problems as estimation problems. By discussing students' worksheets—who used estimation strategies—we expected that other students would learn and realize that the problems are estimation problems.

- For Problem 11, we expected that students would recognize the problem as an estimation problem because they have had an experience in the previous lesson (Problems 9 and 10). Therefore, they were expected to solve the problem by estimation strategies.
- For Problem 12, we expected that students would solve the problem by estimation strategies with help of using their knowledge—i.e. students should know a maximum number of seats in a bus, for example. Next, they were expected to reflect their answers whether these were reasonable or not.

5.2 Retrospective analysis: Research period July-August 2008

In this section, like in section 4.3 (Chapter 4), we focus on the analysis of students' estimation strategies in solving estimation problems. The analysis is specifically focused on students' estimation strategies, students' difficulties in solving problems, kind of problems which invites students to use estimation, and kind of learning-teaching situations that invites students to use estimation. Based on the HLT 2, in short, we expected that students would increasingly use estimation strategies.

We first analyze strategies used by students to solve estimation problems. Similar to the results of the first research period, in the second period students also used two kinds of strategies: estimation strategies (EST) and an exact calculation strategy (EXA), as summarized in Table 5.3. Estimation strategies which were also used by students are *rounding* and *front-end* strategy, where *rounding* strategy was used most. This means, in this case, the cognitive processes used by students belong to *reformulation*. The difference is, here, strikingly all students did not use any estimation strategy to solve estimation problems with incomplete or unavailable data (Problems 9 – 12)—we discuss this later on in this section. We ask the same question as in section 4.3 (Chapter 4): why did students use only *reformulation*—instead of *translation* or *compensation*?

In addition to answers of the same question in the previous chapter, from analysis of video recordings and field notes we found that the teacher did not guide the students to use other cognitive processes (*translation or compensation*) rather than *reformulation* to solve estimation problems. A possible explanation of this can be the following: it might be possible that the teacher herself does not know about other estimation strategies beyond *rounding* or *front-end* strategy; it might be because the other two cognitive processes (*translation or compensation*) are beyond school mathematics curriculum for primary students, which might mean these two cognitive processes are difficult to be reached by most of primary school students, therefore the teacher did not try to use these in the lessons.

Table 5.3: Students' actual strategies in solving estimation problems (July-August 2008)

Problems	Students' actual strategies		
	% EST	% EXA	% U
1	26	72	2
2	23	70	7
3	11	51	38
4	27	22	51
5	79	13	8
6	42	47	11
7	58	33	9
8	58	20	22
9	0	100	0
10	0	100	0
11	0	100	0
12	0	100	0

Note: EST = Estimation strategies; EXA = Exact calculation strategy; U = Unclear.

Based on the analysis of the research results of period May-June 2008 in section 4.3 and the analysis above, **as a conclusion, we summarize possible reasons why students only use two kinds of estimation strategies: *rounding* and *front-end* strategy—which belong to *reformulation*.** First, the problems used during the two research periods did not invite students clearly to use other cognitive processes (*translation* and *compensation*). Second, it might be possible that *translation* and *compensation* are too difficult for most of primary school students of grade four or five therefore it is not given in the mathematics school curriculum. Third, from the video and field notes analysis, during the lessons, the teacher did not use other cognitive processes to solve estimation problems rather than *reformulation*, therefore students used only *rounding* or *front-end* strategy.

In a similar manner to the analysis of the first research period, we now present an overview of overall percentages of students using estimation during the second research period by considering the graph in Figure 5.1.

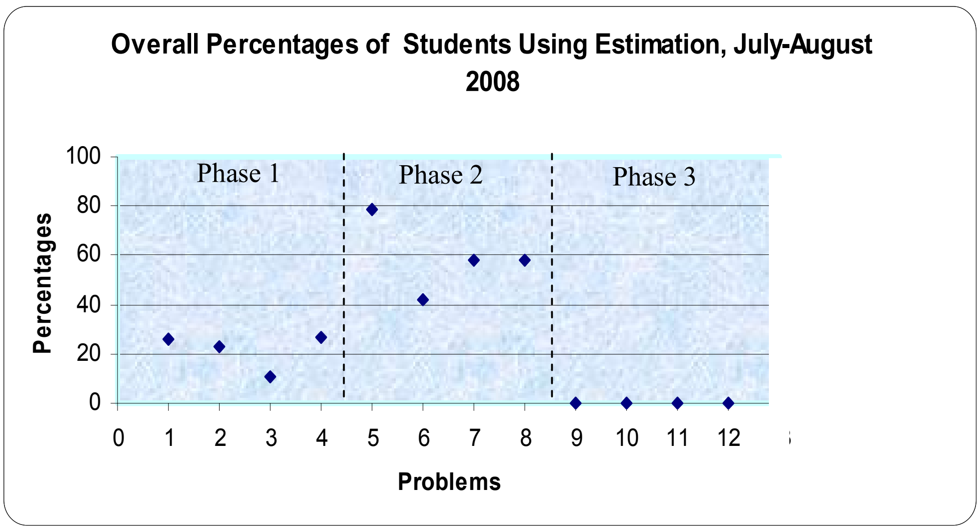


Figure 5.1: Overall percentages of students using estimation in the period July-August 2008

From the graph (in Figure 5.1), for Problems 1 to 8 except for Problem 5, we might see an upward trend in the use of estimation strategies as we expected in HLT 2, however, interestingly after Problem 8 none of the students used an estimation strategy.

In the HLT 2 we classified problems into three groups: Problems 1 – 4, Problems 5 – 8, and Problems 9 – 12. We in retrospect understand this classification because it can be distinguished into three phases as indicated in the graph of Figure 5.1. In phase 1, the average percentages are around 21 %, which means that around 21% students solved problems using estimation strategies. In phase 2, the average percentages are around 59%, which means that around 59% students solved problems using estimation strategies. However in phase 3, it is very surprising because none of the students used an estimation strategy. This result is absolutely different from the result of the first research period.

From the graph in Figure 5.1, there are at least two observations that need more explanation: (1) Problem 5 evoked a sudden high percentage in the use of estimation; and (2) inkblot problems (Problems 9 – 11) and an unavailable data problem (Problem 12) have very different percentages in the use of estimation if compared with the result of the first research period. For example, around 55 % students used estimation strategies to solve Problem 9 (or Problem 6.a) in the first research period, 0% students used estimation strategies in the second period. What are possible explanations for these observations?

Observation 5.1: Problem 5 evoked a sudden high percentage in the use of estimation

Around 80% of students solved Problem 5 using estimation. Why does this problem invite more students to use estimation strategies? To find reasons, at the start we look at the problem, in Figure 5.2, and its possible solutions strategies.

Problem 5: Consider the figure! Using Rp 20,000; how many small or big ice creams could you buy? Explain your answer!

Figure 5.2: Problem 5 (research period July-August 2008)

In the HLT, we expected that students would quickly think that the prices of the big and the small ice creams are Rp 6,000 and Rp 4,000 respectively because the real prices are close to those prices. Since the question is open, we then expected that students would find different possible strategies and answers. Several possible strategies and all possible answers to Problem 5 are as follows.

- If one only wants to buy the big ice creams, then she/he will get 3 ice creams because 3 x Rp 6,000 = Rp 18,000 which is close enough to Rp 20,000.

- If one wants to buy two big ice creams, then she/he will also get two small ice creams, because 2 x Rp 6,000 + 2 x Rp 4,000 = Rp 12,000 + Rp 8,000 = Rp 20,000, which is of course the real price is less than Rp 20,000.
- If one wants to buy only one big ice cream, then she/he will also get 3 small ice creams, because Rp 6,000 + 3 x Rp 4,000 = Rp 6,000 + Rp 12,000 = Rp 18,000 which is close enough to Rp 20,000.
- If one wants to buy only small ice creams, then she/he will get 5 small ice creams because 5 x Rp 4,000 = Rp 20,000 which is of course less than the real cost Rp 20,000.

Other possible estimation strategies that might be used are as follows.

- One would divide Rp 20,000 by Rp 6,000 to obtain 3 and a remainder, which means she/he would get 3 big ice creams.
- One would divide Rp 20,000 by Rp 4,000 to obtain 5 which means she/he would get 5 small ice creams.
- One would divide Rp 20,000 by Rp 10,000 (which is equal to Rp 6,000 + Rp 4,000) to obtain 2, which means she/he would get 2 both for small and big ice creams.

Of course, there might be still other different estimation strategies. For students who did not see this problem as an estimation problem, we predicted that they would solve the problem by an exact calculation strategy, where all possible answers are the same as above.

The following are possible reasons why this problem invited more students to use estimation strategies:

- We see that the problem itself is an open problem. This might invite students to find different answers and strategies. Therefore estimation strategies will be used by most of students because rounding off numbers in this problem is easy (for example, rounding off 5,950 to 6,000 is easy since these are close). Consequently, in this case, estimation strategies are easier than an exact calculation strategy.
- We see operations of numbers that are used to solve the problem include: addition, multiplication, division, or a combination of these three. However, the calculation is relatively easy and flexible: for example most of the students can use only addition (repeated addition) to find answers—this gives an opportunity for students who still find difficulties in multiplication or division to find right answers.
- Other reasons can be found from video analysis—when the class was discussing Problem 5. First we look at the teacher's role during the lesson that might influence the students to use estimation in solving this problem.

- In introduction of the lesson, the teacher introduced the topic of ice cream problems. During the introduction, students were paying attention enthusiastically and were seemingly understanding to the ice cream context, as found in a video transcription below:

  ***Teacher*:** *[Holding a stack of worksheets] Ok students, in this worksheets there is a picture of [one of] your favorite foods.*

  ***Students*:** *Hurray… [The students enthusiastically react to their teacher]*
  [Then, one of students says: "(Is that) banana?" Another student says: "Ice cream?"]

  ***Teacher*:** *Yes, that is right!…. [Now, for today] in this worksheet there is a problem about ice creams. Mmm… so when you buy ice creams you should look at the prices to prepare whether your money is enough or not. In this worksheet you should solve the ice cream problem [the teacher distributes the worksheets to students]*

  As a consequence, most of students might grasp the problem directly; they would see the problem like what happened in their real life; and it might also happen that students would not recognize the problem as a mathematical problem, which usually needs exact answers. This indicates that the ice cream problem context is experientially real for students—this reflects one of the tenets of RME.

- When students were solving the problem individually, the teacher told them many times that there would be many possible answers; and she also emphasized to students frequently not to be afraid if they made mistakes.

  ***Teacher*:** *For Problem [5], you use only Rp 20,000, it could be possible among you [students] have different answers.*
  [Most of students are paying attention to the teacher's explanation]

***Teacher*:** *[For example] one of you would like only buy 1 big ice cream and the other are small ice creams. Other students could have different options…*

[Most of students are still just paying attention]

***Teacher*:** *And you [should be confident], feel not afraid to [if you] make a mistake…*

From video analysis and field notes, we noted that the teacher did not say like this in the previous lessons. Hence, we think that this condition could: make students be confident with their own answers even though their friends have different answers—this reflects one of the tenets of RME; encourage students to find answers with their own strategies; guide students to find different strategies. Therefore, students would use different estimation strategies to find different answers.

- During group discussions, we found students shared their strategies among their friends. We also see most of the students grasp the problem. As an example we found, in the following dialogue, how a student found all possible answers using estimation strategies.

***Researcher*:** *For Problem [5], what do you think if you only buy the small ice creams?*

***Student*:** *If I only buy the small one, I will get 5 ice creams, because each price is Rp 4,000.*

***Researcher*:** *[How many ice cream would you get] if you only buy the big ones?*

***Student*:** *It will then remain Rp 2,000 [from Rp 20,000], I will only get 3 big ice creams because its each price is Rp 6,000 [Rp 6,000 x 3 = Rp 18,000]*

***Researcher*:** *But why in your worksheet you wrote that you will buy 2 big and 2 small ice creams?*

***Student*:** *To make a balance! 2 big and 2 small ice creams!*

***Researcher*:** *Why?*

***Student*:** *Because according to the problem, I asked to buy how many big or small ice creams! [Although this is not fully a right reason]*

- From field notes we can also find other possible reasons that might influence students in solving the problem.
  - The lesson, which discussed Problem 5, took place in the first and second hour of the school day (it is usually in third and four hour of school day. However, since in the first and second hour the teacher of another subject was absent, and the mathematics teacher was ready, so the time was used for mathematics lesson). Therefore, we think most of the students were still fresh: they can concentrate in the lesson better. Besides that, during discussion with the teacher before the lesson, she promised to the researcher to use the same context as problems in worksheets for the introduction; she also understood about the possibilities of different answers. This happened because the researcher asked her to use the same context as the problem in the students' worksheet (because in the previous lessons, the teacher used different context from the problems). This might mean the teacher was well prepared better than previous lessons.
  - Because the teacher was given a worksheet (contain this problem) few days before the lesson, the teacher might have made a better preparation than before. She might have read and learned the problem before the lesson and she might think that the problem is fit with her approach in giving the lesson. Therefore, she performed well in guiding students to solve the problems. For other lessons, she actually was also given worksheets, but we did not know whether she learned them well or not. We guess that she might less pay attention to the given worksheets because when we had discussions before each lesson, she was frequently eager to discuss about problems in the worksheets.

- It might be possible that students had sufficient experience in the use of estimation after solving Problems 1-4 in the previous lessons. As examples of students' answers for Problem 5, we can see Figures 5.3 – 5.6.

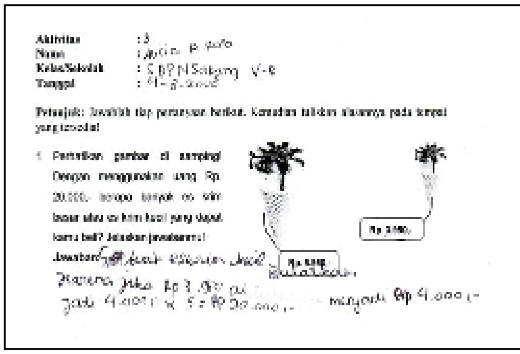


Figure 5.3: Alicia's answer to Problem 5, using estimation
Translation: 5 small ice creams. Because if Rp 3,950 is rounded off to Rp 4,000, then 4000 x 5 = Rp 20,000

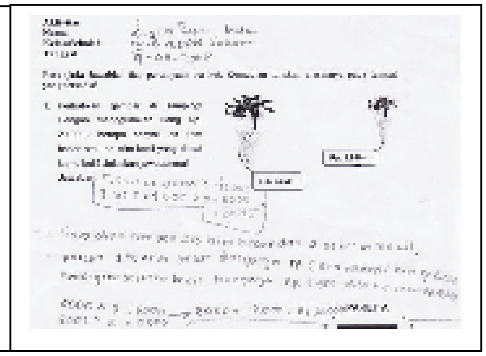


Figure 5.4: Fajrin's answer to Problem 5, using estimation
Translation: I will buy 2 big and 2 small ice creams. Because 1 big ice cream is Rp 5,950, it is rounded off to Rp 6,000. While 1 small ice cream is Rp 3,950, it is rounded off to Rp 4,000. 4,000 x 2 = 8000 and 6000 x 2 = 12,000, therefore, 8,000 + 12,000 = Rp. 20,000.

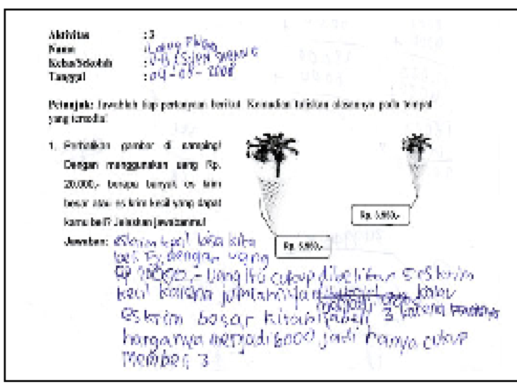


Figure 5.5: Lativa's answer to Problem 5, using estimation
Translation: We can buy 5 small ice creams using Rp 20,000. This money is enough because 5 ice creams are enough using that money [4000 x5]. While we can buy 3 big ice creams because the price [Rp] 6,000 [x 3 = Rp 20,000]

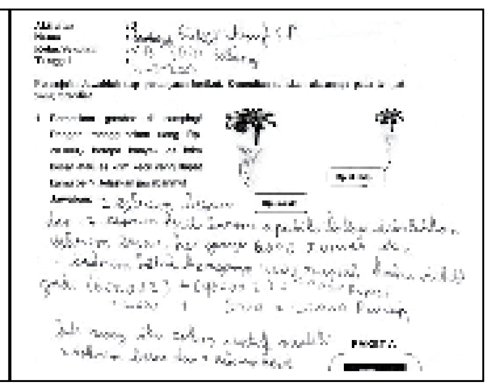


Figure 5.6: Yusuf's answer to Problem 5, using estimation
Translation: 2 big and 2 small ice creams. Because if the prices are rounded off then the price of big and small ice creams are Rp 6,000, and Rp 4,000. Therefore, (6,000 x 2) + (4,000 x 2) = 20,000 or 12,000 + 8,000 = 20,000. So, our money is enough to buy 2 big and 2 small ice creams.

Thus, particularly in our research, we can conclude that Problem 5 is good to invite more students both in the use of (different) estimation

strategies and in producing various answers. Furthermore, we could say that this problem can be an example of good estimation problems that can be used either for learning-teaching estimation or for future research.

Observation 5.2: No estimation used to solve Problems 9 –12

Here we focus on analysis of the first two inkblot problems (Problems 9 – 10) by comparing results of the first and second research period.

The results, in the use of estimation, of Problems 9 – 10 are interesting: in the first research period these two problems—previously Problems 6.a and 7.a respectively—were relatively successfully done using estimation strategies, namely 55% and 50% students solved these by estimation strategies respectively. However, it is very surprising in the second research period none of students used an estimation strategy. What are possible explanations for this observation? To find possible explanations we look at the problems themselves, classroom cultures, and classroom discourse and the teacher behavior.

We at first present the problems, in Figures 5.7 and 5.8, and its possible solution strategies below.

```
   28●                 9●2
   4●● +               489 --
A. 62●              A. 4●●
B. 7●●              B. 56●
C. 557              C. 6●●
```

Figure 5.7: Problem 9

Figure 5.8: Problem 10

In the HLT, for Problem 9, students were expected to solve the problem using one of the possible estimation strategies below.

- To add 28● and 4 ●● students might look at the front digits and rounding off these. Hence, 28● is seen as 200; and 4●● as 400. Therefore, 200 + 400 = 600. Consequently, it might be possible that students would choose the option **A** as the answer, although this is not true. However, for students who see 28● as 300 (rounding to the nearest hundred), they would find that the best possible answer is **B** because 300 + 400 = 700.
- It might be possible that students would use front-end strategy. Students would see 28● as 2 (200) and 4●● as 4 (400), then add 2 + 4 = 6 (600). Next, we expected they would look at the options. Hence, they would see directly that option **C** is impossible. And the

other two are possible. If students stop until this step, they might find that **A** is the best possible answer to the problem. Whereas, for students who, then, see the second digit of 28●, they would see that the addition at least would be 680, so option **A** is impossible. Consequently, we expected they would choose **B** as the best possible right answer.

It might be possible that students would not recognize the problem as an estimation problem. Hence, we predicted they would use an algorithm for addition. Thus, to add 28● and 4●● students would first add from the right side, namely ● + ●, then 8 + ●, and finally 2 + 4 = 6 (or 7). If this last addition is 6, then the result of addition is 68●. (There is no option). Therefore, students will choose **B** as the answer.

In the HLT, for Problem 10, in Figure 5.8, we expected that students would use one of possible estimation strategies below.

- To subtract 9●2 by 489, students might look 9●2 as 900 (rounding off to the nearest hundred) and 489 as 500. Hence, 900 – 500 = 400. But, students might also round off 9●2 to 1000 and 489 to 500, so 1000 – 500 = 500. Therefore, there are two possible right options, **A** and **B**. However, **B** is impossible because 489 + 56● > 1000. Consequently, the best possible right answer is the option **A**.
- Students might use front-end strategy by seeing 9●2 as 9 (900) and 489 as 4 (400), then do a subtraction 9 – 4 = 5. Hence, option **C** is impossible. Students who stop until this thinking might choose **B** as the best possible right answer, although this is not true. But students who see that 56●, then add 56● + 489 which are more than 1000 would choose A as the best possible right answer.

Students who do not see the problem as an estimation problem might solve the problem using an algorithm for subtraction (addition). Namely, they would subtract from the right side: 2 – 9, then ● – 8 and finally 9 – 4. Again, by using similar arguments as the possible estimation strategies above, students would find a possible right answer.

First we look at the problems. Because some numerals are covered by inkblots, then students (of the second research period) might find it difficult to see the problems as estimation problems. Instead, they might think to find possible numerals to replace the inkblots to do common addition and subtraction. Therefore, they would use an exact (trial and error) calculation strategy. However, this reason might not convince readers because in the first research period, most of other students did differently: they used estimation strategies.

Second we look at the situation when students were solving the problems. In the first research period, students worked totally individually. There were no group and class discussion and there was also no teacher guidance. Because of these conditions, we predict that students had the freedom, without any external intervention, in solving the problems. As a consequence, most of the students had found that the inkblot problems are a kind of estimation problems. Therefore, they use estimation strategies. On the other hand, in the second research period, although at first the students worked individually, next the students did group and class discussion with also guidance from the teacher. This means there were external interventions that can influence the students to solve the problems. Consequently, the students had no entire freedom in solving the problems. We found from the analysis that the teacher gave intervention: by giving examples how to solve the inkblot problems using exact (trial and error) calculation strategy, as we found in the video transcription below.

In the beginning of the class: the teacher wrote the problem like in Figure 5.7. Next she tells students how to solve it, like in the following:

Teacher: *We see here, you can read by your own, you solve this inkblot problem by replacing each inkblot with any number.*
[Students still do not understand, so the teacher repeats what she told previously]

Teacher: *You could replace the inkblots by choosing any number. And this can happen between you and your fiends would get different numbers.*

Student: *Ooo… so the numbers are arbitrary.*

From students' mathematical background, we can also find a reason. In the first research period, we worked with a PMRI class, where the students are used to solving mathematical problems by their own strategies, which reflects one of the tenets of RME. Therefore, students might have confidence using their own thought to solve the inkblot problems. Consequently, they solved the problems using estimation strategies. In the second research period, however, we worked with a non-PMRI class, where the students are not used to solve mathematical problems by their own strategies, and students still have dependence on their teacher when solving mathematical problems. This is why in the previous paragraph we predicted that students were possibly influenced by their teacher's intervention.

There might be a fourth reason. We predict that students of the non-PMRI class are used to solving addition or subtraction problem using an algorithm only, therefore, they might not think about the positional system of numbers. Instead of working from the left to the right (which means considering the magnitudes of numbers), when doing addition or subtraction, they might do algorithmically the other way round: from the right to the left. Consequently, for example, when solving the addition 28 ● + 4●● students will work from the right side: doing addition ●.+●, then 8 +●, and so forth. Therefore, to make the addition is as easy like as usual, students would replace the ●. with a number, for example, with zero, like in Figure 5.10 or other numbers like in Figure 5.12. This strategy, replacing the inkblots with zeros, according to students is very easy, as found in the interview below.

Interviewer: *Let me know how did you solve Problem 9?*
Hannan: *[She reads the problem in her worksheet] I think the most possible answer is B!*
Interviewer: *Why?*
Hannan: *[She confused…] Because….*
Interviewer: *Why did you not choose A?*
Hannan: *Because it is wrong!*
Interviewer: *Why is it wrong?*
Hannan: *Mmmm… because when I was adding these* [28● + 4●●] *the results is not A!*
Interviewer: *Why did you choose zero to replace the blanks? [The interviewer points to Hannan's worksheet]*
Hannan: *Because that number is the easiest!*

Finally, we can look at the lesson preparation. Before the lesson, we discussed with the teacher about solution strategies for the inkblot problems. We found that the teacher herself did not recognize the inkblot problems as a kind of estimation problems. She then solved the problems using an exact (trial and error) calculation strategy, like in Figures 5.10 and 5.12. She argued that this strategy is an effective strategy to solve the problems, and she also thought that her strategy is a kind of estimation strategies. Although we had discussed that the inkblot problems could be solved using estimation strategies, not using strategy that she had proposed, but we found, from video analysis, during the lesson the teacher gave examples to students how to solve the problems using an exact (trial and error) calculation strategy. Therefore, students might be influenced by their teacher intervention and, hence, they follow the teacher's strategy to solve the inkblot problems.

Examples of students' answers of Problems 9 and 10 both from the first and second research period can be seen in Figures 5.9 – 5.12 below.

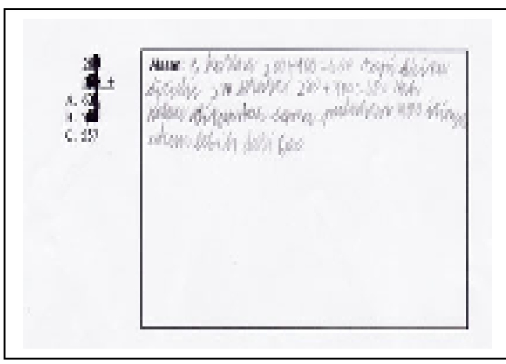


Figure 5.9: Jodi's answer to Problem 9 (research May-June 2008)
Translation: B because 200 + 400 = 600, but it is written 28●, which can mean 280 + 400 = 680. Hence, if we added up [280] and 400, then the sum are more than 600.

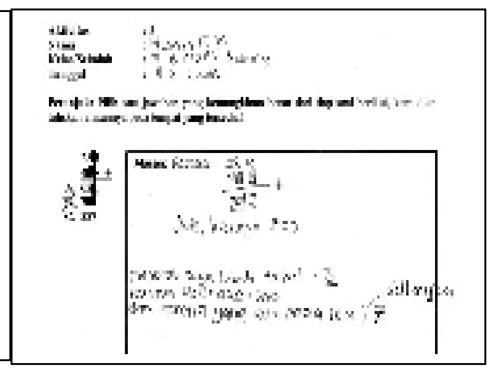


Figure 5.10: Hannan's answer to Problem 9 (research July-August 2008)
Translation: According to me, the most possible right answer is B because 280 + 420 = 700 and in the option [B] there is a number 7.

Figure 5.11: Jodi's answer to Problem 10 (research May-June 2008)
Translation: A, because 900 – 400 = 500. And this result is still not subtracted yet by its tens [of 489], so the result should be less than 500.

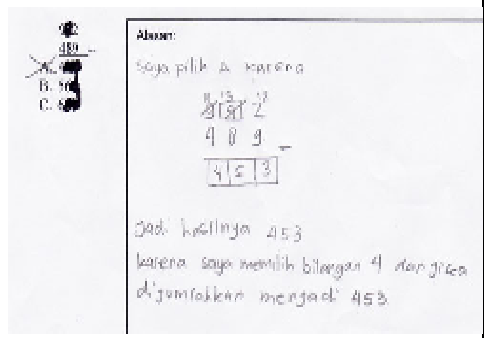


Figure 5.12: Amirah's answer to Problem 10 (research July-August 2008)
Translation: I choose A because 942 – 489 = 453. So, the result is 453. Because I choose a number 4, then if 942 is subtracted [by 489] it becomes 453.

Thus, for revising the HLT, to invite students in the use of estimation to solve the inkblot problems (especially the problems 9 and 10), the teacher should not give too much intervention by giving examples how to solve the problems—using an exact (trial and error) strategy. In addition, giving an understanding to students about positional system of numbers, when working with addition or subtraction problems, might be possible to catch students to use estimation to solve inkblot addition or subtraction problems.

Next, in order to answer the fourth research question: *What kind of learning-teaching situations invite students to use estimation?* We look at teaching-learning situations that happened in the teaching experiment. First, we discuss in general what happened, then we give two lessons scene as examples how the learning-teaching situations happened.

General description of learning-teaching processes on estimation

In the teaching experiment we used the non-PMRI class of the first semester of grade five (10 – 11 years old), however, the teacher tried to use a RME approach, where the students had never been taught using this approach previously. Before the teaching experiment, the researcher discussed with the teacher a plan how the research would be implemented. In addition, before each lesson, the teacher discussed with the researcher how to stimulate particular learning-teaching processes, how to solve estimation problems, and how to manage different possible answers of the problems.

How did the teacher organize the lesson in the classroom? From the analysis of the field notes and video recordings we can observe a general pattern for each lesson. First, the teacher introduced a topic that would be discussed. The introduction was set out from daily activities which were experientially real for students. While the teacher was explaining the introduction, the students were paying attention to her to perceive the topic. This part took 5 to 10 minutes. Second, students solved problems individually and wrote their answers on worksheets. After working for 20-25 minutes, the students discuss with their friends in their group. In group discussion most of the students share strategies in solving the problems (which each student has his/her own worksheet). While students were working either alone or in groups, the teacher was observing students from one group to other and was available to help. The students had the freedom to ask the help. This part took 15-20 minutes. Third, after the students were ready, a class discussion was held. The teacher selected two or three students from different groups to present their worksheets in front of the class. Frequently students presented different strategies for the same problems. The teacher guided how the class discussion proceeded. Mostly students would agree to their friend's presentation, even though sometimes they disagree. Finally, after all problems' answers were presented the class was finished.

With the framework of Wood, Williams, and McNeal (2006) this kind of classroom culture could be characterized as a combination of three: problem solving, strategy reporting, and inquiry arguments classroom cultures. In problem solving classroom culture, students are solving non-routine problems and text-book problems. In strategy reporting, students present different strategies for the problems solved. And in the inquiry argument, students present different solution methods and give reasons why they did that so that other students understand.

However—also from video recordings and field notes analysis—the actual learning-teaching implementation above has differences with the plan in HLT 2.

When introducing lessons, the teacher did not always use the same contexts as used in the estimation problems. Instead, the teacher tried to use different contexts that have to do with the problems. This is perhaps because the teacher thought that to guide students in solving estimation problems they should know other contexts before solving the problems in the worksheets. As a consequence, we think, most of the students would not directly perceive the problems in the worksheets. Besides that, in our view, the teacher sometimes gave too much guidance, for example, telling the students how to solve problems. As a result, most of students solved the problems by the teacher's strategy, which means they did not really solve the problems on their own. Although students sat in groups, sometimes most of them solved the problems individually. This means they did not really share estimation strategies. During the class discussion, many times most of the students presented their solutions but the other students paid less attention to them. In addition, most of the students presented their answers with a low voice. Hence, the teacher was forced to repeat what they had presented. As a consequence, other students who paid attention might only receive what the teacher told—as repetitions of students' presentations—to them without any disagreement. Therefore, many class discussions, can be said, were not very interactive.

Thus, from the description above, we may conclude that although the teacher tried to implement learning-teaching situations based on the plan (using the RME approach), but in reality she did not fully succeed. Consequently, these conditions influenced students' learning processes which indicated by students' results in the second research period.

However, during the second research period, the lessons were not always unsuccessful there were also learning-teaching situations that can be said successful. From the video recordings and field notes analysis, we give two examples: a successful one and unsuccessful one consecutively below.

(1) Classroom learning-teaching situation in lesson 3 (Problems 5 and 6)

We speculate the classroom learning-teaching situation of lesson 3 is successful because the results show that 79% and 42% students solved the

Problems 5 and 6 by estimation strategies respectively (see Table 5.3). In this example, we focus on how the learning-teaching situation has influenced the students to use estimation to solve Problem 5 because this problem is the most successful solved by estimation strategies.

In the introduction, the teacher set out the lesson with the ice cream context. This is done to make students perceive the problems. This context is the same as the context used in the problems that would be discussed. While the students were paying attention to the teacher, it seemed that they could grasp the context easily. This is because, we think, the students are familiar to the ice creams, which means that the ice cream context is experientially real for students (see a dialogue between the teacher and students in Observation 1 of this section).

After explaining the context of problems, the teacher asked the students to solve problems individually to elicit their own strategies. Many times, when students were working individually, the teacher emphasized that they would have different answers, and they should not feel afraid if made mistakes (see a dialogue between the teacher and students when they were working individually in Observation 1, of this section). Furthermore, during this time, the teacher was available to help her students who found difficulties. She walked around in the class from one to another student to give guidance. Besides that when the students were working, the teacher said that one can present his/her answers in front of the class if he/she has different strategies than the other, as found in the video transcription below.

Teacher: *If you have different opinion and different answers with your friends, you can present your work in front of the class.*
[The students are paying attention, although they are still working]

Teacher: *Mmm… at least if you have different answers you can present it! Of course with the reasons [why did you arrive at different answers? Not only answers]. And do not feel afraid! [If you made mistakes]*

We think this emphasis has motivated students to find different answers or different strategies to solve the problems.

Further, the students were working in groups (3 or 4 students each group). In group discussions we see among students shared different opinion, strategies, explanation, etc, interactively. We think this encouraged the students to explain and justify their strategies each other. In addition, during group discussions, the teacher required that who had never given a presentation in front of the class should take turn then.

Teacher: *Ok, please you discuss [your strategies in your groups]! I want you who never gave presentation to come in front of the class [to give a presentation]*

This, we think, could encourage more students to come up with different strategies and different answers, so they could give presentations.

Next, in the class discussion, several students gave presentations: two of them are as follows.

Student 1: *If we have Rp 20,000, then we could buy 2 big and 2 small ice creams. [She takes a breath. Other students are paying attention because before the Student 1 gives a presentation, the teacher read the problem aloud]*

Student 1: *The price of a big ice cream is Rp 5,950, it is rounded off to Rp 6,000, so Rp 6,000 x 2 = Rp 12,000. The price of a small ice cream is Rp 3,950, it is rounded off to Rp 4,000. So, Rp 4,000 x 2 = Rp 8,000. Therefore, Rp 12,000 + Rp 8,000 = Rp 20,000. Accordingly, the money [Rp 20,000] is enough.*

Teacher: *[After giving praise to the student after the presentation] Do you [all other students] understand with your friend's presentation? [Most of the students say understand, the other just keep silent]*

After that, another student gave a presentation, but he presented same answers as the Student 1's. Therefore, the teacher asked others who had different answers or different strategies to give presentations. However, many students were still shy at that time because they were not used to giving presentations in front of the class. But then there was another student (Student 2) came to the front of the class.

Student 2: *[With very low voice she started the presentation. So, other students ask her to talk loudly]…Yeah with the Rp 20,000, we can buy the ice creams. The price of a big ice cream is Rp 5,950, it is rounded off to Rp 6,000; and the [price of a] small one Rp 3,950 is rounded off to Rp 4,000.*
[While she is giving her presentation, many other students are not paying attention. So, the teacher asks the students to keep silent]

Student 2: *So, if we want to buy only the big ice creams, we will get 3 ice creams [because Rp 6,000 x 3 = Rp 18,000 is almost Rp 20,000]. And, if we want to buy only the small ice creams, we will get 5 ice creams [because Rp 4,000 x 5 = Rp 20,000]*

Next, other students give presentations of another problem. Finally, after presentations, the class was finished. Before finishing the class, the teacher gave a review of the learning-teaching processes. For example, the teacher repeated the Student 2's presentation because she gave different answers than the other students (whose answers: 2 big and 2 small ice creams].

As a conclusion, from the learning-teaching situation above, we see two important factors that made students were successful to use more estimation strategies in solving Problem 5, namely: the role of the teacher and the lesson structure of classroom learning-teaching situation—where the former is the most important (see also the conclusion for lesson 5). We see the students would pay most attention to the teacher and would follow what she told either during group or class discussion. The lesson structure,

particularly groups and class discussion, might have motivated the students to share opinion, justification, and strategies among them.

(2) Classroom learning-teaching situation in lesson 5 (Problems 9 and 10)

We assume that the classroom learning-teaching situation of lesson 5 is unsuccessful because the results show none of the students solved Problems 9 and 10 by estimation strategies. In this example, we focus on how the learning-teaching situation had influenced the students not to use estimation to solve Problem 9. How did the learning-teaching situation of the lesson 5 happen?

After distributing worksheets to students (in the introduction) the teacher wrote down Problem 9 like in Figure 5.13 below.

```
     28…
     4…..  +
  A  62…
  B  7…..
  C  557
```

Figure 5.13: Problem 9 (from a video analysis)

The teacher then explained to the students how to solve it (see a dialogue in Observation 2 of this section), namely by replacing the inkblots with arbitrary numbers. Further, the teacher emphasized that the answers would be different because they depend on the choices of numbers. After the introduction, the students worked individually.

After that, in group discussions, there were many students who did not understand the problem. Therefore, they asked to the teacher to explain again, as found in the video transcription below.

Teacher: *[Using Problem 9 that she wrote down in the board, like the Figure 5.13] When you answer the problem, first you may choose the options [A, B, or C], then you give the reasons.*
[The students are just paying attention to the teacher]

Teacher: *The reasons, for example, because you choose arbitrary numbers to replace the inkblots, then doing addition, and so forth. This implies there would many different answers [this means because any arbitrary numbers can replace the inkblots]. So, you may have more than one answer.*

However, there were still students who did not understand after this explanation. For example, a student asked his friend to re-explain what the teacher had explained.

Student A: *How should I solve the problem? [Because I still do not understand the teacher's explanation]*

Student B: *[According to the teacher]. We could choose any number [to replace the inkblots]*

Student A: *Arbitrary? [He is still not sure, maybe because he still does not understand yet]*

***Student B**:* *Yes!*

Student A: *Then, what should I do? [After choosing arbitrary numbers]*

[The Student B just keeps silent, he does not answer the question from the Student A]

To prepare a class discussion, the teacher asked to students whether they were ready or not. Many students were ready which can be seen because they rose up their hands. Next, several students present their answers on the board. The following, in Figure 5.14, are students' answers on the board.

$$\begin{array}{lllll} & 2 & 8 & \boxed{9} & \\ & 4 & \boxed{4} & \boxed{5} & + \\ \hline \text{A} & 6 & 2 & \boxed{} & \\ \text{B} & 7 & \boxed{3} & \boxed{4} & \\ \text{C} & 5 & 5 & 7 & \end{array} \qquad \begin{array}{llll} 2 & 8 & 0 & \\ 4 & 4 & 0 & + \\ \hline 7 & 2 & 0 & \end{array} \qquad \begin{array}{llll} {\scriptstyle 1} & {\scriptstyle 1} & & \\ 2 & 8 & 8 & \\ 4 & 4 & 4 & + \\ \hline 7 & 3 & 2 & \end{array}$$

Figure 5.14: Three different students' answers of Problem 9 [written in the board]

In the end of the lesson, the teacher reviewed what they had learned. She misunderstood about the meaning of the inkblot problems as a kind of estimation problems, as found in the following transcription.

Teacher: *So, today actually we have learned how to estimate numbers that are still unknown [What she understood was actually 'estimation' not for a number as a whole, but a numeral as part of a number]*

Thus, based on the learning-teaching situation above, we see that the most important factor which made students did not use estimation strategies to solve Problem 9 is the role of the teacher: telling how to solve the problem by an exact (trial and error) calculation strategy. This implies the students used the exact (trial and error) strategy to follow the teacher's strategy.

5. 3 Proposal to revise a hypothetical learning trajectory on estimation

Based on the analysis of both research periods above, to revise the HLT 2 for inviting more students to use estimation, we should take into account the following factors:

- Mathematical problem themselves

This includes: kind of problems, operation of numbers, and type of numbers. Kind of problems is meant as a classification of problems: whether the problems are complete (all information is given), incomplete, or unavailable data. Based on the research results of the second research period, we found that estimation problems with incomplete or unavailable data are generally more difficult than problems with complete data (see section 5.2). This result is in line with the proposal from Van den Heuvel-Panhuizen (2001)—that is estimation problems with incomplete or unavailable data are generally more difficult than problems with complete data. Thus, for the purpose of learning-teaching we should give problems with complete data first.

Operation of numbers means mathematical operations which are used for solving problems: addition, subtraction, multiplication, or division.

Mathematically seen, for the same type, magnitudes of numbers, and context, estimation problems in addition or subtraction are easier than in multiplication or division. Thus, for the purpose of learning-teaching we should give problems with addition or subtraction first.

Type of numbers can mean: integer numbers, rational numbers, real numbers, etc. In general, problems which include only integers are relatively easier than problems with fractions or decimal numbers (rational numbers) or even combinations of these. Consequently, for learning-teaching purposes, we should give problems which include integer numbers first, then problems with other types or combinations of these.

- Design of problems

This includes: difficulties, context of problems, openness of problems, selection of numbers, and type of questions.

Difficulties of problems mean the number of solution steps of problems. A problem which needs more than one step solution is generally more difficult than another problem which only needs one step. In our research, we can take an example: Problem 9.a and 9.b (see Table 4.1) are difficult for students because these need more than one step solution. Thus, for a revision of the HLT 2, we should take into account at number of problem step solutions.

Regarding context of problems, we think that to get students to use estimation, we should design problems with contextual situations that are experientially real for students, fit with the students' age, or fit with the students' world. Therefore, the problems can be perceived by them. For example, in our research, the ice cream context is best fit for students because they are used to facing such problems in their daily life. On the other hand, supermarket context (like a receipt context) is less experientially real for students—especially in Indonesia. Because supermarket problems are usually faced by adults: especially in Indonesia, the supermarket context more suitable to mothers. Thus, for a revision of our designed problems, we might need to look for other contexts that are really experientially real for students to replace several problems that are not really fit.

To revise the HLT, we should also consider about the openness of the problems. In our example, the ice cream problem is an open problem with different answers and strategies. Therefore, the problem can invite students to use more different estimation strategies. Thus, we need to change several problems, from closed problems to open problems—with careful considerations, for example, either from designer problems experts or from experienced teachers.

The selection of numbers means that numbers are used in the problems should be close to nearest tens, hundreds, thousands, or other easy numbers because such numbers are able to invite more students to use estimation. For example, we can see Problems 5 – 8 (see Table 5.2), where these problems used numbers near to tens, hundreds, thousands. In practice, for example, rather than using a number like 1,675 it is better to use 1,950 to catch students to rounding off the number to 2,000. Thus, in the case of our problems, we need to revise several numbers in the Problems 1 – 4 (see Table 5.2) to numbers that are close to nearest tens, hundreds, or thousands.

According to Van den-Heuvel Panhuizen (2001), three types of questions that can be used to invite students to use estimation strategies are*: Are there enough*? (2) *Could this be correct?* (3) *Approximately how much* is *it?* In the case of our problems, in this research, we used only the first and the second type of questions. Thus, to revise the HLT 2, we need to add problems with the third type of questions.

- Classroom cultures

This includes: students' own productions and strategies, teacher guidance, group and classroom discussion (interactivity in learning-teaching situations). Referring to the tenets of RME, students should be guided in explorations and solving problems of contextual problems within an interactive classroom. In our case, we tried to implement the tenets of RME and its principles; however it was not really attained. In the case of Problems 9 – 12, in our view the teacher has given too much intervention, namely giving examples how to solve the problems by an exact calculation and giving clues how to find solutions of inkblot and unavailable data problems.

In the case of Problem 5 that is relatively successful in inviting more students to use estimation, we see from the video recordings and field notes analysis (can be found in section 5.2), the teachers' role and the lesson structure of classroom learning-teaching situation are very important.

Thus, to revise the HLT, we should prepare the teacher better to make sure whether the teacher understands in giving guidance to his/her students or not. In case the teacher wants to give help, she/he should understand entirely the topic that will be taught during the learning-teaching processes. Thus, a good teacher's preparation before lessons is important. Besides that, classroom learning-teaching structure in the form of group and class discussion might motivate students to share: opinion, justification, and strategies. Therefore, a good classroom management is very important to be considered by teachers.

Therefore, the HLT should include not only instructional materials but also teachers' preparation explicitly (on understanding the topics, problems, materials, and classroom management).

6 Conclusion and discussion

The aims of this research were to investigate students' strategies in solving computational estimation problems and to gain insight into how students can be stimulated to use estimation. In the light of these aims, we conducted design research with the following research questions:

1. *What strategies do students use to solve estimation problems?*
2. *What are students' difficulties in solving estimation problems?*
3. *What kind of problems invite students to use estimation?*
4. *What kind of learning-teaching situations invite students to use estimation?*

The answer to the first and second research question can contribute to an understanding of students' strategies in solving computational estimation problems, while answers to the third and the fourth can contribute to gaining insight into how students can be stimulated to use estimation in solving computational estimation problems.

6.1 Answer to the first research question

To answer the first research question, we summarize the analysis regarding students' strategies from both research periods as follows.

Strategies used by students to solve computational estimation problems from both research periods, as predicted in the HLT, can be classified into two: estimation strategies and exact calculation strategy. The estimation strategies which are used consist only of *rounding* and *front-end* strategies, where the *rounding* strategy is used most. According to Reys et al. (1991) these strategies belong to a cognitive process which is called *reformulation*. Other cognitive processes which did not emerge from students during the research are *translation* and *compensation.*

Possible reasons why students only use *rounding* and *front-end* strategies can be the following. First, the problems used during the two research periods do not invite students clearly to use the other cognitive processes. Second, it might be possible that *translation* and *compensation* are too difficult for most of primary school students of grade four or five, which is why these two cognitive processes are not given in the Indonesian mathematics school curriculum, or it could be the other way round: the school mathematics curriculum does not give place for other cognitive processes than *reformulation*. Finally, based on video recordings and field notes, we conclude that during lessons the teacher did not use other cognitive processes to solve estimation problems than *reformulation*.

Exact calculation strategy that emerged during the research is in the forms of exact calculation by an algorithm and of an exact (trial and error) calculation (see Table 4.3 and Table 5.3 for percentages of students solving computational estimation problems by an exact calculation strategy). In the case of Problems 9 – 12 in the second research period, for example, all students used exact (trial and error) calculation strategy to solve the problems (see Table 5.3).

In particular, to solve Problems 9–10 (the inkblot addition and subtraction problems), students in the second research period used the exact (trial and error) calculation strategy because of the following possible reasons: (1) Students had never encountered such problems before, so they might not have recognized the problems as computational estimation problems rather than just common addition and subtraction problems; (2) Based on video recordings and field notes analysis the teacher gave too much guidance: by consistently showing the exact (trial and error) calculation strategy. Therefore, students followed the teacher's strategy, whereas at the first period there was no teacher to follow; (3) Students of the second research period are not used to solve mathematical problems on their own: they are still depending on the teacher's instruction; this might be because the students are not used to be taught by the RME approach, whereas students in the first period (the PMRI class) are used to be taught by the RME approach; and (4) students might not think about positional system of numbers when solving these problems; In other words, they did not use the magnitudes of numbers (which means they see the numbers from the left to the right). Instead they might always use an algorithm (which means they solve the problems by seeing numbers from the right to the left).

6.2 Answer to the second research question

Based on the analysis of both research periods we categorize students' difficulties into the following:

Difficulties related to numbers

At least three difficulties related to numbers can be observed from students' answers. First, students had difficulties in using numbers that are not clearly factors of other numbers. For example, a problem which includes 10,000 and 5 is easier to be calculated than the problem that includes 15,000 and 7 because 5 is a factor of 10 whereas 7 is not factor of 15. This example can be seen in Problems 4.a and 4.b (see Table 4.1).

Second, students experience difficulties rounding off numbers that are "too" far from the nearest tens, hundreds, thousands, or other easy numbers. For example, the number 1675 is less inviting students to round off than the number 1950 to 2000 because it is 'too' far to the nearest thousand. This example can be seen in the explanation of Problems 8.a or 8.b (see Table 4.1)

Third, most of the students experience difficulties in solving estimation problems that have to do with fractions or decimal numbers. For example, most of the students can not solve Problem 9.a or 9.b (see Table 4.1).

Difficulties related to the complexity of the problems

We also observe at least three difficulties faced by most of the students related to the complexity of the problems. Problems which need more than one step solutions generally are more difficult than problems with one step solution; it is also difficult to translate problems with a lot of information; most students tend to avoid this kind of problems. For example, to solve Problem 9.a or 9.b (see Table 4.1) students need more than one step solution, moreover, it also contains a lot of information.

Most of the students have difficulties with inventing realistic data by themselves to solve problems because they are used to solve problems with all information given: it is difficult to connect various pieces of information from the problems themselves and the outside (knowledge, experience, etc) to find an answer, and it is also difficult to think reflectively what they have done because most of them are used to only do a calculation without much understanding, without enough self-awareness whether the answers or processes are reasonable or not. For example, to solve Problem 13 (see Table 4.1) students should use their experience to produce realistic number of seats in a bus, combine information from the problem with such data to do a calculation and then finally judge whether the answer is reasonable or not.

Finally, most of the students also found it difficult to solve problems that they had never encountered before. This, therefore, is included as problem solving without fixed procedures to solve such problems. To such problems, they are less able to judge whether the problems are computational estimation problems or not. For example, when solving multiplication inkblot problem (see Table 5.2), most of the students do not recognize that the problem is actually as an estimation problem. Instead of solving the multiplication inkblot problem by estimation strategies, they used an exact (trial and error) calculation strategy. In particular, to solve the inkblot problems, we guess that most students might forget to use knowledge of positional system of numbers (or magnitudes of numbers) to find answers.

Difficulties related to students' habits

We think that most of the students are not easy to convince to use estimation strategies in solving computational estimation problems. This is because of the following possible reasons: (1) students have become used to solve mathematical problems with exact answers throughout their school career; (2) mathematics teachers at school usually demand precise answers to mathematical problems. Therefore, when solving computational estimation problems the students tend to find exact answers beside estimate answers; and (3) there is a very small part of mathematics curriculum that addresses computational estimation.

6.3 Answer to the third research question

Based on our experience with the students involved in the research period we speculate that the following characteristics of computational estimation problems can invite more students to use estimation strategies:

- Contexts used in problems should be experientially real for students, so they can immediately grasp the problems. For instance, the ice cream context in Problem 5 (see Table 5.2 and Observation 5.2).
- The problems should elicit various different answers or elicit various different strategies. For example, Problem 5 (Table 5.2).
- The numbers involved in the problems should initially be close enough to the nearest tens, hundreds, thousands, or other easy numbers. Therefore, the numbers can be easily rounded off, which are important for estimation. For example see Problems 5 and 6 (Table 5.2).
- Operations of numbers which are used to solve the problems should be flexible. This means, if possible, we should design problems that can be solved by different operations (addition, subtraction, multiplication or division). This characteristic can give an opportunity to students who find difficulties in doing particular operations to find right answers. For example, students can use 'easy operations than difficult ones to solve problems (for instance, addition than multiplication).
- The questions used in the problems should not require exact answers. Type of questions, like asking whether enough or not, whether correct or not can invite students to use estimation strategies (Van den Heuvel-Panhuizen, 2001).

An example of problems that almost fulfill the characteristics above is Problem 5, in Figure 5.2. This problem indeed invited more students to come up with different strategies and different possible answers.

6.4 Answer to the fourth research question

Based on our experience with students involved in the second research period, we found two aspects that might invite more students to use estimation in solving estimation problems, namely the lesson structure of classroom learning-teaching situation and role of the teacher.

Regarding the lesson structure of the classroom learning-teaching situations, in particular, we could say that group and class discussions might motivate students to share opinion, justification, and (estimation) strategies. In this way, students could learn from each other. This implies there would be interactivity or re-inventing strategies from discussion with other students. However, this might not be better than learning-teaching situation without group or class discussion because we found from the first research period that the students' result are better than the students' results of the second research period for inkblot and estimation problems with unavailable data problems.

Regarding the teachers' role, we found in our case (Indonesian culture) that the students would generally follow what have been explained by the teacher. For example, when the teacher explained a clue to solve Problem 5 (in the second research period) the students followed the teacher's strategy. This might be because in Indonesian culture, the teachers at schools are supposed to be the same as the students' parents (at home); the teachers' explanations are supposed to be very trustworthy. Accordingly, particularly in Indonesian case, 'stronger' teacher is important in establishing a norm. She should stimulate students to think themselves in solving computational estimation problems. In this way, hopefully students would use estimation strategies.

Thus, a good preparation on either mathematics or classroom management before implementing learning-teaching is indispensable for teachers.

6.5 Discussion

In this section we reflect on the research findings. We then think of possible improvements based on the findings either for use in classroom learning-teaching or for future research (ideas for HLT revision).

Students' estimation strategies

As we mentioned in the previous sections, the estimation strategies used by students only include *rounding* and *front-end* strategy. One reason why only those strategies emerged in students' answers is because the problems used in the research do not clearly invite students to use other strategies. Therefore, we think to improve or use other problems. One possible way by changing the nature of numbers involved in the problems—so the problems invite students to use other estimation strategies. Consider an example below.

In Problem 1, students are asked to add the numbers up in Figure 6.1. To invite students use other estimation strategies we can change the numbers and also the goods—to make the problem experientially real—for instance. So, it becomes Figure 6.2. For the case in Figure 6.2, instead of using *rounding* strategy: 4000 + 4000 + 4000 + 4000 + 4000 + 4000 + 4000 + 4000, it is shorter to use *changing operation* strategy: 8 x 4000.

2 @ KANGKUNG	3,750
3 @ BAYAM	4,550
1 KG KOL PUTIH	1,675
1,5 KG AYAM SAYAP	14,000
2 @ INDOMILK	24,800
400 GRAM BAWANG MERAH	2,990
250 GRAM CABE MERAH	4,499
1 BTL COOKING OIL	6,295

Figure 6.1: A receipt clip from Problem 1

Item 1	3,950
Item 2	4,125
Item 3	3,875
Item 4	4,000
Item 5	4,100
Item 6	3,990
Item 7	4,250
Item 8	3,800

Figure 6.2: An example of a revision of Problem 1

Inkblot problems

One possible reason why students find it is difficult to solve inkblot problems is because the students might not think of positional system of numbers. This means that students are used to solve addition, subtraction, and multiplication from the right to the left without necessarily looking at the magnitudes of numbers (work from the left to the right). For example, consider Problems 9 and 11 (see Table 5.2) that are rewritten in Figures 6.3 and 6.4 below.

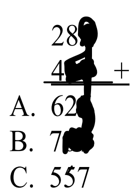


Figure 6.3: Problem 9

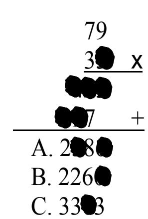


Figure 6.4: Problem 11

How do we improve the inkblot problems in order to make students would be aware of positional system of numbers? We think there are several possible ways to do that, as described in the following.

First, we might simplify the problems by reducing the inkblots. For example, in case of Problem 9 in Figure 6.3 above, we could change 28●. and 4●● for instance, become a column addition 281 + 4●● in Figure 6.5. Therefore, students would concentrate only to one number 4●● to find a possible right answer by considering magnitude of the number to make an estimate. Similarly, in case of Problem 11, we could reduce the inkblots in the problem to be a problem in Figure 6.6.

```
   281
   4●●  +
-----------
A. 62●
B. 7●●
C. 557
```

Figure 6.5: Revision of Problem 9

```
    79
    3● x
---------
A. 2●8●
B. 226●
C. 33●3
```

Figure 6.6: Revision of Problem 11

Second, we could change the format of problems without reducing the inkblots, namely from a column addition to a row addition, from column multiplication to a row multiplication, etc. For example, in case of Problem 9 above, we could change the problem to be 28●. + 4●● =… with the answer options are still available. Therefore, students are expected to think possible magnitudes of the numbers 28●. and 4●●. In a similar manner, Problem 11 becomes 79 x 3●. =…. also with the options are available.

Third, by combining the first and second ways above, namely change the inkblot problems by reducing the inkblots and changing the format. Therefore, in case of Problem 9, we could change it, for example to 281 + 4●● = …. And in case of Problem 11, it becomes 79 x 3●. = ….

Regarding Problem 11, we found interesting students' answers that used rather different strategy, namely excluding impossible options to find an answer. In the analysis we included this strategy as an exact (trial and error) calculation strategy. However, we think this strategy is interesting to be discussed because it is different from the common exact (trial and error) calculation strategy. For example, see a student answer in Figure 6.7.

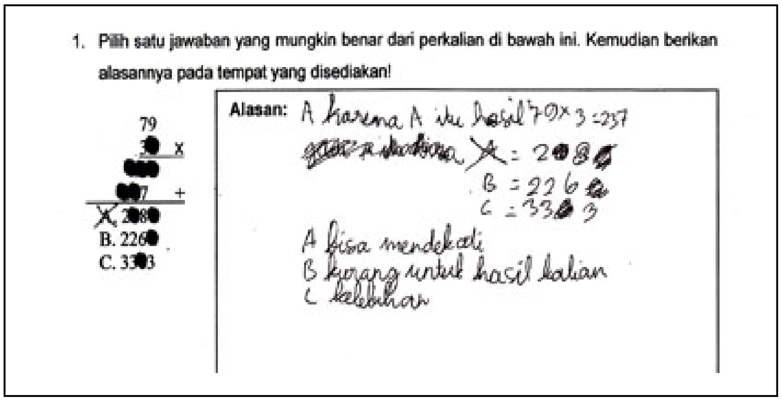


Figure 6.7: Destiana's answer to Problem 11

Translation: **A** because it is from 79 x 3 = 237. **A** is possible because it is close; **B** is less than the result of multiplication [79 x 3]; **C** is more than [the result of multiplication 79 x 3].

We think this strategy is different from the exact (trial and error) strategy because when we are excluding impossible options we use a different cognitive process rather than cognitive processes that happen if we use the exact (trial and error) calculation strategy. After all, we do not know yet what kind of cognitive processes used in this strategy.

Bus problem

We found in the first research period the students were less tempted to solve the bus problem (Problem 13, see Table 4.1) by estimation strategies than an exact calculation strategy. More surprisingly, none of students solved the bus problem (Problem 12, see Table 5.2) by estimation strategies in the second research period.

One possible reason is because most of the students have not been trained to think reflectively: they are only used to doing calculation, without looking back to the calculation results. As a consequence, they would not be aware whether what they did was reasonable or not. We think, in the case of the bus problem, to make students think reflectively we should slightly change the question. Do not ask whether **make sense or not the news** but we change it by a question like, for example, **could the number of buses bring 9998 supporters?** In this way, we predict students would better perceive the problem and estimation strategies hopefully would be used.

Another possible reason is because students are not used to combine information from the problem itself and from outside the problem—in this case real-world knowledge or experience. This might be caused by students' view that mathematics (arithmetic) and real-world contexts are separate systems. Therefore, when students should solve unavailable data problems, for instance the bus problem, they would only concentrate on the problems and might not think to use other information from out side the problems. Thus, in learning-teaching situations, we think teachers should give experiences to students to solve problems that combine information both form the problems themselves and from outside the problems. Moreover, giving rich context problems to students hopefully would change their view: from the view that mathematics and contexts are separate systems to a new view that mathematics and context can be connected.

In the PMRI class (first research period), students might have used to solve contextual problems that combine information from problems themselves and outside the problems, that might explain why there were students solved the bus problem by estimation strategies. In the non-PMRI class (second research period), however, students are not used to solve problems that combine information from the problems themselves and outside the problems, that might explain why none of students use estimation strategies to solve the bus problem.

How to prepare teacher(s) during design research in Indonesian cultures?

One of the three phases in design research is the teaching experiment. As researchers, we should do this phase carefully because the teaching experiment is the core of the design research (Gravemeijer, 2004). During this phase, based on field notes and video recordings, we found difficulties concerning preparation of the teacher as indicated in the following.

- Although before the teaching experiment and before each lesson we have a discussion with the teacher about a plan how to implement the lesson, however, she sometimes did not follow it. For example, in introducing lessons, the teacher sometimes used different contextual situations from the context used in the problems.
- We assumed that the teacher understood the philosophy of RME because she had 3 – 4 years experience, in the PMRI project, using a RME approach in learning teaching situations. However, in our view, the teacher gave too much guidance to students. For example, she told to students how to solve inkblot problems. This implies students did not solve problems on their own instead they followed the teacher's strategy.

Because in Indonesian cultures we should give a great respect to the teacher, we then were reluctant to give suggestions—this is impolite. Moreover, since we are younger than the teacher, we should very appreciate to the teacher's decisions.

Thus, such cultural issues are important to take into account when we try to implement design research in educational practice, particularly in the teaching experiment phase in Indonesia. This also could be a consideration when we will conduct co-design research in the PMRI project, for example.

Teachers' role and classroom cultures' differences between PMRI and RME

RME is developed in the Netherlands and PMRI is the Indonesian version of RME. Although between Indonesia and the Netherlands had a very close connection in the past, they have very different cultures. This might happen also in educational practice. Therefore, PMRI and RME classroom cultures may have differences, as described below.

- Classroom social norms that generally established in Indonesian situations are: students are generally not used to expressing their thinking in front of the class, students are reluctant to ask questions to the teacher if they do not understand yet, students try to avoid different arguments either with the teacher or other students that expressed directly in the class. These imply less interactivity in the classroom learning-teaching situations—although the interactivity is one of the tenets of RME. This could be in contrast if we compare to the Dutch students' characters. As a consequence, if we try to implement a RME approach in Indonesian classroom situations, teachers would have a big challenge in fostering interaction.

- We found from the analysis of video recordings and field notes that the students in general would follow what the teacher had explained to them for granted. This means one classroom social norm that established particularly in the non-PMRI class is that students are dependent on the teacher explanation. This implies that teachers should be careful in giving guidance during learning-teaching situations. As a consequence, an understanding to one of principles of RME in guiding students—namely guided reinvention—is important. This might be different from the Dutch situations, where Dutch students might not follow everything from the teacher's explanation. Our impression is that Dutch students are more critical and perhaps less polite. They seen more used to think for themselves.

Therefore, such potential differences are important to take into account when we try to implement a RME approach in Indonesian classroom situations through PMRI.

Possible future research

It was not until the analysis of our data that we realized that there was a discrepancy between the use of estimation in daily life and the teaching of it in our classrooms: estimation in daily life is mental, without paper and pencil, whereas we allowed students to do estimation with paper and pencil. One way to avoid students using exact calculation is by making estimation more experientially real: let them do mental calculation and oral explanation. Whether they will work in Indonesian context is an interesting topic for future research.

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Appendices

Appendix 1: Comparison between hypothetical learning trajectory and students' actual strategies

Table 8.1: Comparison between HLT 1 and students' actual strategies of research period: May-June 2008

Problems	Prediction(s)	Students' actual strategies	n/N	%
1.a	With this problem**,** students are expected to round off numbers of the prices and add them to know whether the sum is less or more than 50 (50,000). Students are expected to do this because if the prices are added up exactly then it would be difficult and also the question does not require an exact calculation. So, there would be several strategies that might be done by students to add the prices, as the following: - One might solve the problem by adding round numbers of the prices as follows: (3 + 4) + (1 + 14) + (25 + 3) + (4 + 6) = (7 + 15) + 28 + 10 = 22 + 38 = 60, where "3" means "3000", "4" means "4000", etc. Hence, here clearly that the sum is more than 50 (50,000). - One might directly look at the biggest number, then add the remaining number as follows: 25 + 14 + 1 = 25 + 15 = **40**, then **40** + 3 + 4 + 6 = 43 + 10 = **53**. This is more than 50 even the numbers have not been added at all. - One might solve the problem by adding easy numbers (friendly numbers) such as follow: 6 + 4 = **10**, next **10** + 25 = **35**, then **35** + (14 + 1) = 35 + 15 = 50. But since there are more prices to be added, it is clear that the sum of all the prices is more than 50 (50,000). - One might do by grouping easy numbers and add them, as follows: 3 + 4 + 1 = **8**, next **8** + 2 = **10**, and 6 + 4 = **10**, so it is 20. Since 25 + 14 = **39**, then **39** + **20** > 50. - One might solve as follows 3+ 4 = **7**; 1 + 14 = **15**, 4 + 6 = **10**. Thus, **7** + **15** + **10** + **25** = 57 > 50. - Etc. Students who did not see the problem as an estimation problem would	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 11% of the students used estimation strategies.	2/18 14/18 2/18	11 78 11

	solve by an exact calculation strategy.			
2.a	With this problem, students are expected to see this problem as a subtraction problem. However, for students who do not see this problem as a subtraction problem, at least they will solve similar to the Problem 1.a. Therefore, the following are strategies that might be used by students. - For the students who look this problem as a subtraction problem, since the sum of all the prices in the question 1 is more than Rp 50,000, then by subtracting it by the price of milk, namely around to Rp 25,000, they will get Rp 25,000. But since the extra off price is much more, then they will conclude that Rp 25,000 is not enough to buy all the goods except the INDOMILK. - For the students who do not look the problem as a subtraction problem, they might solve by the same strategies as they did in solving the problem 1.a Students who did not see this problem as an estimation problem will solve by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 11% of the students used estimation strategies.	2/18 14/18 2/18	11 78 11
3.a	With this problem, to compare which one is cheaper between the two things, of course the students will look to the prices. However, since the numbers of the prices are complicated, then a calculation by an algorithm would be difficult. Hence, we expected the students will use the following possible strategies. - Since the Rp 3,750 is close to the Rp 3,800, the students might find the price of a bundle of Kangkung is close to 1/2 of Rp 3,800, namely Rp 1,900. Similarly, because Rp 4,550 is close to Rp 4,500, then a bundle Spinach is close to 1/3 of Rp 4,500, namely Rp 1,500. Thus, they can conclude that a bundle of Kangkung is more expensive than a bundle of Spinach. - The students might solve by comparing the prices of, for example, 6 bundles of Kangkung and 6 bundles of Spinach, namely: 6 Kangkung = 3 x Rp 3,750 > 10,000 but 6 Spinach = 2 x Rp 4,550 < 10,000. So, they can conclude that a bundle of Kangkung is more	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 28% of the students used estimation strategies.	5/18 8/18 5/18	28 44 28

	expensive than a bundle of Spinach. - Etc. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.			
1.b	To solve this problem**,** students are expected to round off numbers of the prices and add them to know whether the sum is less or more than 80 (80,000). Students are expected to do this because if the prices are added up exactly then it would be difficult and also the question does not require an exact answer. So, there would be several strategies that might be used by students in adding the prices, as follows. - One might solve the problem by adding round numbers of the prices, namely: (4 + 5) + (2 + 14) + (25 + 3) + (5+ 6) = (9 + 16) + 28 + 11 = 25 + 39 = 64, where "4" means "4,000", "5" means "5,000", etc. This is less than 80 (80,000). - One might directly look at the biggest numbers, then add the remaining rounding numbers: 25 + 14 + 2 = 25 + 16 = **41**; next **41** + 4 + 5 + 6 = 46 + 10 = **56**; and finally **56** + 3 + 5 = **64** < 80. - One might do by grouping easy numbers and add them, as follows: 4 + 5 + 1 = **10**, next **10**+ 14 = **24**, and **24** + 25 = **49**. Since 6 + 4 + 3 = 10 + 3 = **13**, then **49** + **13** = 62 < 80. - Etc. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 18% of the students used estimation strategies.	3/17 13/17 1/17	18 76 6
2.b	With this problem students are expected to see the problem as a subtraction problem. However, for the students who do not look the problem as a subtraction problem, then at least they will do similar to the Problem 1.b. Therefore, the following are strategies that might be used by the students. - For the students who look this problem as a subtraction problem, since the sum of all the prices in the Problem 1.b is greater than Rp 60,000, then by subtracting it by the prices of the INDOMILK and chicken, namely around to Rp 25,000 and Rp 14,000, then they will	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary:	3/17 9/17 5/17	18 53 29

	get prices more than Rp 20,000. Thus, they can conclude that Rp 20,000 is not enough to buy all the goods except the MILK and chicken. - For students who do not look the problem as a subtraction problem, they might solve it by the same strategies as the Problem 1.b.	18% of the students used estimation strategies.		
3.b	With this problem, to compare which is cheaper between the two things, of course the students will look to the prices. However, since numbers of the prices are complicated, so a calculation by an algorithm would be difficult. Thus, we expected the students will use the following possible strategies. - The first strategy that might be used by the students: they first will look for the price of 1 of each thing, then multiplied by five. And finally compare the prices. Since the Rp 3,750 is close to the Rp 3,800, so the students might find the price of a bundle of Kangkung is close to 1/2 of Rp 3,800 = Rp 1,900. Similarly, because Rp 4,550 is close to Rp 4,500, then a bundle of Spinach is close to 1/3 of Rp 4,500 = Rp 1,500. By multiplying each of these prices by five, they will obtain Rp 9,500 and Rp 7,500 respectively. So, they can conclude that the price of 5 bundles of Kangkung is more expensive than the price of 5 bundles of Spinach. - Another strategy could be as follows. Since the question is asking which one is cheaper, the students might solve by comparing the prices of, for example, 6 Kangkung and 6 Spinach, namely: 6 Kangkung = 3 x Rp 3,750 > 10.000 but 6 Spinach = 2 x Rp 4,550 < Rp 10,000. Thus, they can conclude that a bundle Kangkung is more expensive than a bundle of Spinach. As a consequence, 5 bundles of Kangkung is more expensive than 5 bundles of Spinach. - Etc. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.	- Estimation strategy - Exact calculation strategy - Unclear ============================ Summary: 24% of the students used estimation strategies.	4/17 7/17 6/17	24 41 35
4.a	We expected that students will use one of the following possible strategies.	- Estimation strategies	12/22	55

	- To find the prices of 5 bundles of Kangkung, the students might first find the prices of a bundle of Kangkung, then multiply it by 5. So, since 2 bundles of Kangkung are Rp 3,750, it is rounded off to Rp 3,800. Thus a bundle of Kangkung is close to Rp 1,900. Therefore, 5 bundles of Kangkung are close to 5 x Rp1,900 < Rp 10,000. - To find the prices of 5 bundles of Kangkung from the 2 bundles of Kangkung = Rp 3,750 may be done as follows. If 2 bundles of Kangkung are Rp 4,000, then 10 bundles of Kangkung = 5 x 2 bundles of Kangkung = 5 x Rp 4,000 = Rp 20,000. Thus, 5 bundles of Kangkung are less than Rp 10,000 (Rp 20,000 divided by 2). - To find the prices of 5 bundles of Kangkung, the students may first find the prices of a bundle of Kangkung, then multiply it by 5. Since 2 bundles of Kangkung are Rp 3,750, it is rounded off to Rp 4,000, then a bundle of Kangkung is less than to Rp 2,000. Therefore, 5 bundles Kangkung are less than 5 x Rp 2,000 = Rp 10,000. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.	- Exact calculation strategy - Unclear ============================ Summary: 55 % of the students used estimation strategies.	8/22 2/22	36 9
5.a	The students are expected to use one of the following strategies. - Since 3 x Spinach are Rp 4,550, then to find the number of Spinach that can be bought by Rp 15,000, the students might first find the price of a bundle of Spinach. Then Rp 15,000 is divided by the price of a bundle of Spinach. If Rp 4,550 is rounded off to Rp 4,500, then a bundle of Spinach is actually little bit more than Rp 1,500. Since a bundle of Spinach is little more than Rp 1,500, then Rp 15,000 divided by Rp 1,500 is less than 10, which means 9. Therefore, students can conclude that they could buy 9 Spinach! - Since 3 x Spinach are Rp 4,550, then to find the number of Spinach which can be bought by Rp 15,000, is just by doing a repeated addition: 3 x Spinach + 3 x Spinach + 3 x Spinach = 4,550 + 4,550 + 4,550 which is almost 15,000. So, there will be 9 bundles of Spinach that can be bought.	- Estimation strategies - Exact calculation strategy - Unclear ============================ Summary: 50 % of the students used estimation strategies.	11/22 5/22 6/22	50 23 27

	- One might use the information 3 x Spinach = Rp 4,550, then to be Rp 15,000, it should be multiplied by around 3. Therefore, the students can conclude that there are 9 bundles spinach that could be bought. - One might think as follows. Since 3 x Spinach is Rp 4,550, then one may think if 3 x Spinach are Rp 5,000 then to be Rp 15,000, there will be 9 Spinach. - Etc. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.			
4.b	We expected that students will use one of the following possible strategies. a. To find the prices of 7 bundles of Kangkung, the students may first find the price of a bundle of Kangkung, then multiply it by 7. Since 2 bundles of Kangkung are Rp 3,750, it is rounded off to Rp 3,800, then a bundle of Kangkung is close to Rp 1,900. Therefore, 7 bundles of Kangkung are close to 7 x Rp1,900 < Rp 15,000. b. To find the price of 7 bundles of Kangkung from the information 2 bundles of Kangkung = Rp. 3750 might be done as follows. If 2 bundles of Kangkung are Rp 4,000, then a bundle of Kangkung is Rp 2,000. Therefore, 7 bundles of Kangkung = 7 x Rp 2,000 = Rp 14,000 which is less than Rp 15,000. c. Students who did not see the problem as an estimation problem might solve the problem by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 26% of the students used estimation strategies.	5/19 8/19 6/19	26 42 32
5.b	The students are expected to use one of the following strategies. a. Since 3 x Spinach are Rp 4,550, then to find the number of Spinach that can be bought by Rp 10,000 students may first find the price of 1 Spinach, Next Rp 10,000 is divided by the price of 1 spinach. If Rp 4,550 is rounded off to Rp 4,500, then a bundle of Spinach is actually little bit more than Rp 1,500. Since a bundle of Spinach is little more than Rp 1,500, then Rp 10,000 divided by Rp 1,500 is less than 7, which means 6. Therefore, students can conclude that they could buy 6 Spinach!	- Estimation strategies - Exact calculation strategy - Unclear	8/19 3/19 8/19	42 16 42

	b. Since 3 x Spinach are Rp 4,550, then to find the number of Spinach which can be bought by Rp 10,000, is just by doing a repeated addition: 3 x Spinach + 3 x Spinach = Rp 4,550 + Rp 4,550 which is almost Rp 10,000. So, there will be 6 bundles of Spinach that can be bought by Rp 10,000. c. One may use the information 3 x Spinach = Rp 4,550, then to be Rp 10,000, it should be multiplied by around 2. Therefore, the students can conclude that there are 6 bundles of spinach that can be bought buy Rp. 10,000. d. One may think as follows. Since 3 x Spinach are Rp 4,550, she/he may think if 3 x spinach are Rp 5,000 then to be Rp 10,000, there will be 6 spinach. Students who did not see this problem as an estimation problem might solve by an exact calculation strategy.	============================== Summary: 42% of the students used estimation strategies.		
6.a	28[redacted] 4[redacted] + A. 62[redacted] B. 7[redacted] C. 557 Students are expected to use one of the following possible strategies: - To add 28… and 4… … students might look to the front digits and round off 28… to 200 and 4… … to 400, so 200 + 400 = 600. It could be the students will choose the option **A** as the answer. However, for students who see 28… as 300 (rounding to the nearest hundred), they will find that the best possible answer is **B** because 300 + 400 = 700. - It might possible that the students use front-end strategy. Students will see 28…. as 2 (as 200) and 4… … as 4 (400), next they add 2 + 4 = 6 (as 600). By looking at the options, they will see that the option **C** is	- Estimation strategies - Unclear ============================== Summary: 55% of the students used estimation strategies.	11/20 9/20	55 45

	impossible. And the other two are possible. Students who stop until this step might find that **A** is the best possible answer. Whereas, for students who see the second digit of 28…, they will see that the addition at least would be 680, so the option **A** is impossible. Consequently, they will choose **B** as the best possible answer. - It might be possible that students do not see the problem as an estimation problem, so they will use an algorithm to add. Hence, to add 28… and 4… … they will add from the right side, namely blank + blank, then 8 + blank, and finally 2 + 4 = 6 (or 7). If this last addition is 6, then the result of addition is 68…. (There is no choice). So, students will choose **B** as the answer. - Students who did not see this problem as an estimation problem might solve by an exact calculation strategy with algorithm.			
7.a	9[redacted]2 489 -- A. 4[redacted] B. 56[redacted] C. 6[redacted] Students are expected to use one of the following possible strategies: - To subtract 9…2 by 489, students might look 9…2 as 900 (rounding to the nearest hundred), and 489 as 500. Hence, 900 – 500 = 400. But, they might also round off 9…2 to 1000 and 489 to 500, so 1000 – 500 = 500. Therefore, there are two possible options, **A** or **B**. The option **B** is impossible because 489 + 56… > 1000, hence the best possible answer is **A**. - Students might use front-end strategy by looking 9…2 as 9 (as 900) and 489 as 4 (as 400), then subtract 9 – 4 = 5. Hence, the option **C** is impossible. Students who stop till this thinking might choose **B** as the best possible answer. But for students who see that 56…, then add	- Estimation strategies - Exact trial-error calculation strategy - Unclear ============================== Summary: 50% of the students used estimation strategies.	10/20 7/20 3/20	50 35 15

	56… + 489 which is more than 1000, will choose that the option A is the best possible answer. - Students who do not see the problem as an estimation problem might solve by an algorithm for subtraction (addition). Namely, they will subtract from the right side: 2 – 9, next blank – 8 and finally 9 – 4. Again using similar arguments as the possible strategies above, then the students will find that the option A is the best possible answer.			
6.b	3[redacted]5 5[redacted] + A. 1[redacted] B. 96[redacted] C. 7[redacted]7 Students are expected to use one of the following possible strategies: - To add 3…5 and 5… … students might look to the front digits and rounding them. So 3…5 is seen as 300 and 5… … as 500, hence 300 + 500 = 800. But from the options, **A** and **C** are impossible. Consequently the option **B** is the best possible answer. - It might possible that students use front-end strategy. Students will see 3…5 as 3 (as 200) and 5… … as 5 (500), next add 3 + 5 = 8 (as 800). By looking at the options, they will see that the options **A** and **C** are impossible. Consequently the option **B** is the best possible answer. - It might be possible that students do not see the problem as an estimation problem, so they will use an algorithm for addition. So, to add 3…5 and 5… … students will first add from the right side, namely 5 + blank, next blank + blank, and finally 3 + 5 = 8 (or 9). If this last addition is 8, then the result of addition is 8… …. (There is no choice). So, students will choose **B** as the best possible answer.	- Estimation strategies - Exact calculation (trial-error) strategy - Unclear ============================== Summary: 58% of the students used estimation strategies.	11/19 3/19 5/19	58 16 26

7.b	7[redacted]2 23[redacted] – A. 4[redacted]4 B. 6[redacted]4 C. 1[redacted] Students are expected to use one of the following possible strategies - To subtract 7…2 by 23…, the students might look 7…2 as 700 (rounding to the nearest hundred), and 23… as 200, so 700 – 200 = 500. But there is no option. However, the options B and C are impossible. Therefore, the most possible answer is the option **A**. - Students might use front-end strategy by looking 7…2 as 7 (as 700) and 23… as 2 (as 200), then do the subtraction 7 – 2 = 5 (as 500). Therefore, looking at the option, **C** is impossible. There are two remaining possible options, **A** and **B**. If **B** is the best possible answer, then 6…4 can be seen as 6 (as 600). Therefore, 6 + 2 (from 23…), is greater than 7… 2. So, the best possible answer is **A** - Students who do not see the problem as an estimation problem might solve by an algorithm for subtraction (addition). Namely, they will subtract from the right side: 2 – blank, then blank – 3 and finally 7 – 2. Again using similar arguments as the possible strategies above, then they will find that the option **A** is the best possible answer.	**-** Estimation strategies - Exact (trial-error) strategy - Unclear =========================== Summary: 58% of the students used estimation strategies.	11/19 2/19 6/19	58 11 31
8.a	Several strategies that might be used by students are as follows. - Since the question only requires to know whether Rp 10,000 is enough or not to buy 5 kg of white cabbage, and it is known that Rp 1,675, per kg of the cabbage, then we expected that students will estimate the price of 5 kg of cabbage by rounding off the price/kg Rp	**-** Estimation strategies - Exact calculation strategy - Unclear	6/22 14/22 2/22	27 64 9

	1,675 to Rp 2,000, for example. So, students will find that Rp 10,000 is enough to buy 5 kg of white cabbage because Rp 2,000 x 5 = Rp 10,000. - Students might estimate the price of 5 kg of white cabbage by rounding off Rp. 1,675 to other easy numbers, for example to Rp 1,800 and they will do a multiplication Rp 1,800 x 5 = Rp. 9.000 < Rp 10,000. Thus, they can conclude that Rp 10,000 is enough to buy 5 kg of white cabbage. - Students who do not see the problem as an estimation problem, might solve the problem by an exact calculation strategy: since 1 kg of white cabbage is Rp 1,675, then 5 kg of it is Rp 1,675 x 5 = Rp 8,375 < Rp 10,000. Hence, they can conclude that Rp 10,000 is enough to buy 5 kg of white cabbage.	============================ Summary: 27% of the students used estimation strategies.		
9.a	Since the price of 1,5 kg of the chicken is Rp 14,000, then to find the price of 1/2 kg of it is by dividing the price by 3. Because it is rather difficult (not handy), the question does not require an exact answer, then students will be invited to use estimation strategies. Strategies that might be used by students are as follows. - Since the weight of the chicken is 1,5 kg, and the price is Rp 14,000, then it will be easy if the price is rounded off to Rp 15,000. Hence, the price of 1/2 kg of chicken is 1/3 of Rp15,000 = Rp 5,000. But this is more than the real price of the ½ kg of chicken. So, it is enough using Rp 5,000 to buy 1/2 kg of chicken. - Since 1,5 kg of chicken is Rp 14,000, then 3 kg of it is Rp 28,000 < Rp. 30,000. Hence, one kg of chicken is less than Rp 10,000. This means 1/2 kg of chicken is less than Rp 5,000. - If one kg of chicken is Rp 10,000, then 1/2 kg of it is Rp 5,000, and 1,5 kg of it is Rp 15,000. But the real price of 1,5 kg of the chicken is Rp 14.000 < Rp 15,000. Thus, it is enough to use Rp 5,000 to buy 1/2 kg of the chicken. - If 1/2 kg of chicken is Rp 5,000, then 1,5 kg of it is Rp 15,000 > Rp 14,000. So, it is enough to buy 1/2 kg of chicken by Rp 5,000.	- Estimation strategies - Exact calculation strategy - Unclear ============================ Summary: 18% of the students used estimation strategies.	4/22 1/22 17/22	18 5 77

	- Since 1,5 kg of the chicken is Rp 14,000, then 1/2 of 1,5 kg (or 3/4 kg) of it is Rp 7,000. It is easy to think the price of 3/4 kg of the chicken as Rp 7,500. So, 1/4 kg of it is Rp 2,500. This means 1/2 kg of the chicken is Rp 5,000, where this is more than the real price. - Etc. Students, who did not see the problem as an estimation problem, might solve the problem as follows. First, they will find the price of 0,5 kg by dividing Rp. 14,000 by 3, namely Rp 14,000 **:** 3 = Rp 4.666,7. Hence, they can conclude that Rp 5,000 is enough to buy 1/2 kg of chicken.			
8.b	Several strategies that might be used by students are as follows. - Since the question only requires to know whether Rp 8,000 is enough or not to buy 4 kg of white cabbage, and it is known that Rp1, 675 per kg of the cabbage, then we expected that students will estimate the price of 4 kg of white cabbage, by rounding off the price/kg Rp 1,675 to Rp 2,000, for example. Hence, the students will find that Rp 8,000 is enough to buy 4 kg of cabbage because Rp 2000 x 4 = Rp 8,000. - Students might estimate the price of 4 kg of white cabbage by rounding off Rp 1,675 to other easy numbers, for example to Rp 1,800, next they will do a multiplication Rp 1,800 x 4 = Rp 7,200 < Rp 8,000. Thus, they can conclude that Rp 8,000 is enough to buy 4 kg of white cabbage. - Students who do not see the problem as an estimation problem might solve the problem by an exact calculation strategy: Since 1 kg of white cabbage is Rp 1,675, then 4 kg of it is Rp 1,675 x 4 = Rp 6,700 < Rp 8,000. Hence, they can conclude that Rp 8,000 is enough to buy 4 kg of white cabbage.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 21% of the students used estimation strategies.	4/19 11/19 4/19	21 58 21
9.b	Since the price of 1,5 kg of the chicken is Rp 14,000, then to find the price of 1 kg: first students might find the price of 1/2 kg, namely by dividing the price by 3. However, since it is difficult (not handy), and the	- Estimation strategies - Unclear	4/19 15/19	21 79

	question does not require exact answer, then they will be invited to use estimation strategies. Strategies that might be used by students are as follows. - Since the weight of the chicken is 1,5 kg, and the price of it is Rp 14,000, then it will be easy if the price is rounded off to Rp 15,000. So, the price of 1/2 kg of the chicken is 1/3 of Rp 15,000 = Rp 5,000. Thus, the price of 1 kg of the chicken is Rp 5,000 x 2 = Rp 10,000. But this is more than the real price of 1 kg of chicken. So, it is enough using Rp 10, 000 to buy 1 kg of chicken. - Since 1,5 kg of chicken is Rp 14,000, then 3 kg of it is Rp 28,000 < Rp 30,000. So, one kg of chicken is less than Rp 10,000. - If one kg of chicken is Rp 10,000, then 1/2 kg of it is Rp 5,000, and 1,5 kg of it is Rp 15,000. But the real price of 1,5 kg of the chicken is Rp14,000 < Rp 15,000. Thus, it is enough to use Rp 10, 000 to buy 1 kg of chicken. - If 1/2 kg of chicken is Rp 5,000, then 1,5 kg of it is Rp 15,000 > Rp 14,000. So, it is enough to buy 1 kg of chicken by Rp 10, 000. - Since 1,5 kg of the chicken is Rp 14,000, then 1/2 of 1,5 kg (or 3/4kg) of chicken is Rp 7,000. It is easy to think the price of 3/4 kg of chicken as Rp 7,500. Hence, 1/4 kg oc chicken is Rp 2,500. This means 1 kg of the chicken is Rp 10,000, where this is more than the real price. Students, who did not see this problem as an estimation problem, might solve the problem as follows. First, they will find the price of 0,5 kg of the chicken by dividing Rp 14,000 by 3, namely Rp 14,000 : 3 = Rp 4.666,7, next they will do a multiplication Rp 4.666,7 x 2 = Rp 9333,4. This is less than Rp 10,000. So, they can conclude that Rp 10,000 is enough to buy 1 kg of chicken.	============================ Summary: 21% of the students used estimation strategies.		
10	We expected that students will think that the prices of the big and the small ice creams are close to Rp 6,000 and Rp 4,000. Since the question is open, we then expected that students will have different possible	- Estimation strategies - Exact calculation strategy	15/42 21/42	36 50

	answers, as the following. - If one wants to buy the big ice creams only, then he/she will get 3 ice creams because 3 x Rp 6,000 = Rp 18,000, which is close enough to Rp 20,000. - If one wants to buy two big ice creams, then he/she will also get two small ice creams, because 2 x Rp 6,000 + 2 x Rp 4,000 = Rp 12,000 + Rp 8,000 = Rp. 20,000. - If one wants buy one big ice creams, then he/she will get also 3 small ice creams because Rp 6,000 + 3 x Rp 4,000 = Rp 6,000 + Rp 12,000 = Rp 18,000. - If one wants buy the small ice creams only, then he/she will get 5 small ice creams because 5 x Rp 4,000 = Rp 20,000. Students who did not see the problem in the question 1 as an estimation problem are predicted that they will solve the question using an exact calculation strategy (where all possible solutions are same as above).	- Unclear ============================== Summary: 36% of the students used estimation strategies.	6/42	14
11	We expected that students will think that the prices of the packets A and B are Rp 70,000 and Rp 100,000 respectively because the real prices are close to those prices. We then expected, to compare which packet is the cheaper one, that students will use one of the following strategies. - Students will first try to find the price of each unit of hats. For the packet A, the unit price is Rp 70,000 : 2 = Rp 35,000. While for the packet B, the unit price is Rp 100,000 : 3 < Rp. 34. 000,- Therefore, the students can conclude that the packet B is cheaper than the packet A. - Students might compare, which packet is the cheaper one, by finding the prices of same number of units, for example, students will compare 6 units from the packet A and 6 units from the packet B. Thus, they will not do a division problem, instead they do a multiplication. So, for the packet A, the price for 6 units is 3 x Rp 70,000 = Rp 210,000, while the price for 6 unit of the packet B is 2 x Rp 100,000 = Rp 200,000. Therefore, the students can conclude that the packet B is cheaper than the packet A.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 33% of the students used estimation strategies.	14/42 15/42 13/42	33 36 31

	Students who did not see the problem as an estimation problem might solve the question by an exact calculation strategy.			
12	79 3● x ●●● ●●7 + A. 2●8● B. 226● C. 33●3 We expected that students will see 79 x 3● as an estimation problem, where it can be estimated as a multiplication 80 x 30 or 80 x 40. So, the multiplication 79 x 3● has values between around 80 x 30 = 2400 and around 80 x 40 = 3200. So, the option B and C are impossible (because 79 x 30 = 2370). Therefore, the most possible answer is the option A. Students who did not see the problem as an estimation problem might solve the problem using an exact calculation strategy. For the multiplication above, the students will make trial and error to substitute the inkblot. For example, they will do a multiplication 79 x 30, 79 x 31, 79 x 32, … or 79 x 39. Then, they might choose a possible right answer from the options based on their calculation.	- Estimation strategies - Exact (trial and error) calculation strategy - Unclear ============================= Summary: 21% of the students used estimation strategies.	8/39 10/39 21/39	21 26 53
13	Noticing to the numbers of supporters and buses, then students are expected to round them to be 10.000 and 20 respectively. Therefore, each bus will bring around 500 passengers (supporters). However, we expected the students to use their knowledge or experience that the maximum total passengers in a bus is between 40 and 60 passengers. Therefore, they can conclude that the news does not make sense. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy: 9998 **:** 19 = 526 or 527. Using their knowledge or experience, they can conclude that the news	- Estimation strategies - Exact calculation strategy - Unclear ============================= Summary: 5% of the students used estimation	2/39 14/39 23/39	5 36 59

	does not make sense. Students, who did not realize about the maximum number of passengers in a bus, only doing a computation for example, might conclude that the news makes sense.	strategies.		
14	From the list price, we expected that students will round off the prices per kilogram of apples, oranges and grapes to Rp 12,000; Rp 10,000; and Rp. 20,000 respectively, because all those prices are close to the nearest thousands. Therefore, the students are expected to solve problem using one of possible strategies below. - 1 kg of apples + $\frac{1}{2}$ kg of oranges = Rp 12,000 + $\frac{1}{2}$ x Rp 10,000 = Rp 12,000 + Rp 5,000 = Rp 17,000. Thus, the students can conclude that they have **enough** money to buy 1 kg of apples and $\frac{1}{2}$ kg of oranges by Rp 20,000. Students who did not see the problem as an estimation problem might solve the problem by an exact calculation strategy: - 1 kg of apples + $\frac{1}{2}$ kg of oranges = Rp 11,900 + $\frac{1}{2}$ x Rp 9,900 = Rp 11,900 + Rp 4,950 = Rp 16,850. Thus, the students will conclude that they have **enough** money to buy 1 kg of apples and $\frac{1}{2}$ kg of oranges by Rp 20,000. There might also students who first solve the problem by an exact calculation strategy, next they finally make a rounding off the final result of their calculation to the nearest thousands. For example, after doing the exact calculation and finding the result Rp 16, 850, they will round off this result to Rp 17,000.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 42% of the students used estimation strategies.	15/36 11/36 10/36	42 31 27

15	From to the list prices, we expected the students will round off the prices of each kilogram of apples, oranges and grapes to Rp 12,000; Rp 10,000; and Rp 20,000 respectively because all those prices are close to the nearest mentioned thousands. So, to answer problem, the students are expected to solve the problem by the following strategy. - $\frac{1}{2}$ kg of apples + $\frac{3}{4}$ kg of grapes = $\frac{1}{2}$ x Rp 12,000 + $\frac{3}{4}$ x Rp 20,000 = Rp 6,000 + Rp 15,000 = Rp 21,000. Thus, the students can conclude that they have **enough** money to buy $\frac{1}{2}$ kg of apples and $\frac{3}{4}$ kg of grapes by Rp 25,000. Students who did not see the problem as an estimation problem might solve the problem by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear ============================== Summary: 42% of the students used estimation strategies.	15/36 4/36 17/36	42 11 47

Note: n = number of students used a kind of strategies; N = number of all students

Table 8.2: Comparison between HLT 2 and students' actual strategies of research period July-August 2008

Problems	Predictions	Students' actual strategies	n/N	%
1	With this problem, students are expected to round off numbers of the prices and add them to know whether the sum is less or more than 50 (50,000). Students are expected to do this because if the prices are added up exactly then it would be difficult and also the question does not require an exact calculation. So, there would be several strategies that might be done by students to add the prices, as the following: - One might solve the problem by adding round numbers of the prices as follows: (3 + 4) + (1 + 14) + (25 + 3) + (4 + 6) = (7 + 15) + 28 + 10 = 22 + 38 = 60, where "3" means "3000", "4" means "4000", etc. Hence, here clearly that the sum is more than 50 (50,000). - One might directly look at the biggest number, then add the remaining number as follows: 25 + 14 + 1 = 25 + 15 = **40**, then **40** + 3 + 4 + 6 = 43 + 10 = **53**. This is more than 50 even the numbers have not been added at all. - One might solve the problem by adding easy numbers (friendly numbers) such as follow: 6 + 4 = **10**, next **10** + 25 = **35**, then **35** + (14 + 1) = 35 + 15 = 50. But since there are more prices to be added, it is clear that the sum of all the prices is more than 50 (50,000). - One might do by grouping easy numbers and add them, as follows: 3 + 4 + 1 = **8**, next **8** + 2 = **10**, and 6 + 4 = **10**, so it is 20. Since 25 + 14 = **39**, then **39** + **20** > 50. - One might solve as follows 3+ 4 = **7**; 1 + 14 = **15**, 4 + 6 = **10**. Thus, **7** + **15** + **10** + **25** = 57 > 50. - Etc. Students who did not see the problem as an estimation problem would solve by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear ========================== Summary: 26% of the students used estimation strategies.	10/39 28/39 1/39	26 72 2
2	Several strategies that might be used by students are as follows. - Since the question only requires to know whether Rp 10,000 is	- Estimation strategies - Exact Calculation Strategy	9/39 27/39	23 70

	enough or not to buy 5 kg of white cabbage, and it is known that Rp 1,675, per kg of the cabbage, then we expected that students will estimate the price of 5 kg of cabbage by rounding off the price/kg Rp 1,675 to Rp 2,000, for example. So, students will find that Rp 10,000 is enough to buy 5 kg of white cabbage because Rp 2,000 x 5 = Rp 10,000. - Students might estimate the price of 5 kg of white cabbage by rounding off Rp. 1,675 to other easy numbers, for example to Rp 1,800 and they will do a multiplication Rp 1,800 x 5 = Rp. 9.000 < Rp 10,000. Thus, they can conclude that Rp 10,000 is enough to buy 5 kg of white cabbage. - Students who do not see the problem as an estimation problem, might solve the problem by an exact calculation strategy: since 1 kg of white cabbage is Rp 1,675, then 5 kg of it is Rp 1,675 x 5 = Rp 8,375 < Rp 10,000. Hence, they can conclude that Rp 10,000 is enough to buy 5 kg of white cabbage.	- Unclear ========================== Summary: 23% of the students used estimation strategies.	3/39	7
3	With this problem, to compare which one is cheaper between the two things, of course the students will look to the prices. However, since the numbers of the prices are complicated, then a calculation by an algorithm would be difficult. Hence, we expected the students will use the following possible strategies. - Since the Rp 3,750 is close to the Rp 3,800, the students might find the price of a bundle of Kangkung is close to 1/2 of Rp 3,800, namely Rp 1,900. Similarly, because Rp 4,550 is close to Rp 4,500, then a bundle Spinach is close to 1/3 of Rp 4,500, namely Rp 1,500. Thus, they can conclude that a bundle of Kangkung is more expensive than a bundle of Spinach. - The students might solve by comparing the prices of, for example, 6 bundles of Kangkung and 6 bundles of Spinach, namely: 6 Kangkung = 3 x Rp 3,750 > 10,000 but 6 Spinach = 2 x Rp 4,550 < 10,000. So, they can conclude that a bundle of Kangkung is more expensive than a bundle of Spinach.	- Estimation strategies - Exact calculation strategy - Unclear ========================== Summary: 11% of the students used estimation strategies.	4/37 19/37 14/37	11 51 38

	- Etc. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.			
4	We expected that students will use one of the following possible strategies. - To find the prices of 5 bundles of Kangkung, the students might first find the prices of a bundle of Kangkung, then multiply it by 5. So, since 2 bundles of Kangkung are Rp 3,750, it is rounded off to Rp 3,800. Thus a bundle of Kangkung is close to Rp 1,900. Therefore, 5 bundles of Kangkung are close to 5 x Rp1,900 < Rp 10,000. - To find the prices of 5 bundles of Kangkung from the 2 bundles of Kangkung = Rp 3,750 may be done as follows. If 2 bundles of Kangkung are Rp 4,000, then 10 bundles of Kangkung = 5 x 2 bundles of Kangkung = 5 x Rp 4,000 = Rp 20,000. Thus, 5 bundles of Kangkung are less than Rp 10,000 (Rp 20,000 divided by 2). - To find the prices of 5 bundles of Kangkung, the students may first find the prices of a bundle of Kangkung, then multiply it by 5. Since 2 bundles of Kangkung are Rp 3,750, it is rounded off to Rp 4,000, then a bundle of Kangkung is less than to Rp 2,000. Therefore, 5 bundles Kangkung are less than 5 x Rp 2,000 = Rp 10,000. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear ============================ Summary: 27% of the students used estimation strategies.	10/37 8/37 19/37	27 22 51
5	We expected that students will think that the prices of the big and the small ice creams are close to Rp 6,000 and Rp 4,000. Since the question is open, we then expected that students will have different possible answers, as the following. - If one wants to buy the big ice creams only, then he/she will get 3 ice creams because 3 x Rp 6,000 = Rp 18,000, which is close enough to Rp 20,000. - If one wants to buy two big ice creams, then he/she will also get two small ice creams, because 2 x Rp 6,000 + 2 x Rp 4,000 = Rp 12,000 + Rp 8,000 = Rp. 20,000.	- Estimation strategies - Exact calculation strategy - Unclear ============================ Summary: 79% of the students used estimation strategies.	30/38 5/38 3/38	79 13 8

	- If one wants buy one big ice creams, then he/she will get also 3 small ice creams because Rp 6,000 + 3 x Rp 4,000 = Rp 6,000 + Rp 12,000 = Rp 18,000. - If one wants buy the small ice creams only, then he/she will get 5 small ice creams because 5 x Rp 4,000 = Rp 20,000. Students who did not see the problem in the question 1 as an estimation problem are predicted that they will solve the question using an exact calculation strategy (where all possible solutions are same as above).			
6	We expected that students will think that the prices of the packets A and B are Rp 70,000 and Rp 100,000 respectively because the real prices are close to those prices. We then expected, to compare which packet is the cheaper one, that students will use one of the following strategies. - Students will first try to find the price of each unit of hats. For the packet A, the unit price is Rp 70,000 **:** 2 = Rp 35,000. While for the packet B, the unit price is Rp 100,000 **:** 3 < Rp. 34. 000,- Therefore, the students can conclude that the packet B is cheaper than the packet A. - Students might compare, which packet is the cheaper one, by finding the prices of same number of units, for example, students will compare 6 units from the packet A and 6 units from the packet B. Thus, they will not do a division problem, instead they do a multiplication. So, for the packet A, the price for 6 units is 3 x Rp 70,000 = Rp 210,000, while the price for 6 unit of the packet B is 2 x Rp 100,000 = Rp 200,000. Therefore, the students can conclude that the packet B is cheaper than the packet A. Students who did not see the problem as an estimation problem might solve the question by an exact calculation strategy.	- Estimation strategies - Exact calculation strategy - Unclear =========================== Summary: 42% of the students used estimation strategies.	16/38 18/38 4/38	42 47 11
7	From the list price, we expected that students will round off the prices per kilogram of apples, oranges and grapes to Rp 12,000; Rp 10,000; and Rp. 20,000 respectively, because all those prices are close to the nearest thousands. Therefore, the students are expected to solve problem using one of possible strategies below.	- Estimation strategies - Exact calculation strategy	21/36 12/36	58 33

	- 1 kg of apples + $\frac{1}{2}$ kg of oranges = Rp 12,000 + $\frac{1}{2}$ x Rp 10,000 = Rp 12,000 + Rp 5,000 = Rp 17,000. Thus, the students can conclude that they have **enough** money to buy 1 kg of apples and $\frac{1}{2}$ kg of oranges by Rp 20,000. Students who did not see the problem as an estimation problem might solve the problem by an exact calculation strategy: - 1 kg of apples + $\frac{1}{2}$ kg of oranges = Rp 11,900 + $\frac{1}{2}$ x Rp 9,900 = Rp 11,900 + Rp 4,950 = Rp 16,850. Thus, the students will conclude that they have **enough** money to buy 1 kg of apples and $\frac{1}{2}$ kg of oranges by Rp 20,000. There might also students who first solve the problem by an exact calculation strategy, next they finally make a rounding off the final result of their calculation to the nearest thousands. For example, after doing the exact calculation and finding the result Rp 16, 850, they will round off this result to Rp 17,000.	- Unclear ========================== Summary: 58% of the students used estimation strategies.	3/36	9
8	From to the list prices, we expected the students will round off the prices of each kilogram of apples, oranges and grapes to Rp 12,000; Rp 10,000; and Rp 20,000 respectively because all those prices are close to the nearest mentioned thousands. So, to answer problem, the students are expected to solve the problem by the following strategy. - $\frac{1}{2}$ kg of apples + $\frac{3}{4}$ kg of grapes = $\frac{1}{2}$ x Rp 12,000 + $\frac{3}{4}$ x Rp 20,000 = Rp 6,000 + Rp 15,000 = Rp 21,000. Thus, the students can conclude that they have **enough** money to buy $\frac{1}{2}$ kg of apples and $\frac{3}{4}$ kg of grapes by Rp 25,000. Students who did not see the problem as an estimation problem	- Estimation strategies - Exact calculation strategy - Unclear ========================== Summary: 58% of the students used estimation strategies.	21/36 7/36 8/36	58 20 22

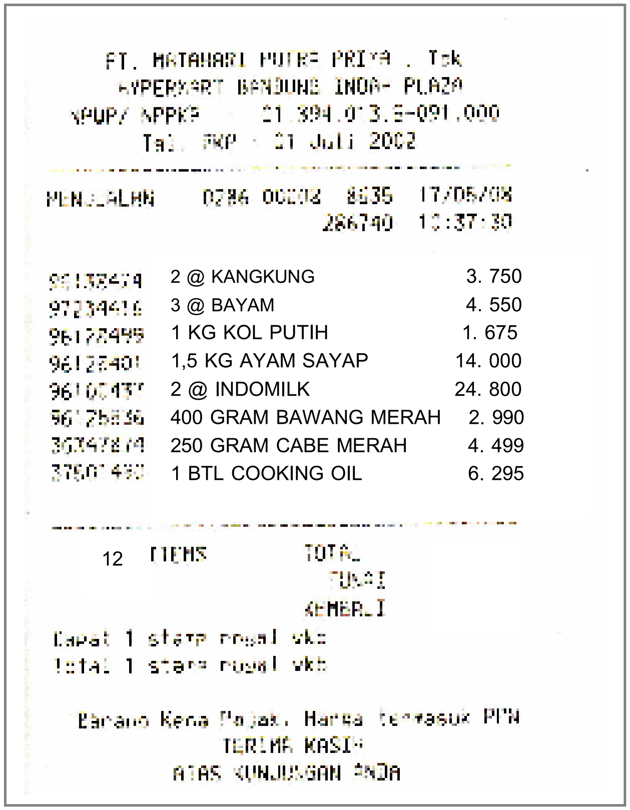
	might solve the problem by an exact calculation strategy.			
9	28[redacted] 4[redacted] + A. 62[redacted] B. 7[redacted] C. 557 Students are expected to use one of the following possible strategies: - To add 28… and 4… … students might look to the front digits and round off 28… to 200 and 4… … to 400, so 200 + 400 = 600. It could be the students will choose the option **A** as the answer. However, for students who see 28… as 300 (rounding to the nearest hundred), they will find that the best possible answer is **B** because 300 + 400 = 700. - It might possible that the students use front-end strategy. Students will see 28…. as 2 (as 200) and 4… … as 4 (400), next they add 2 + 4 = 6 (as 600). By looking at the options, they will see that the option **C** is impossible. And the other two are possible. Students who stop until this step might find that **A** is the best possible answer. Whereas, for students who see the second digit of 28…, they will see that the addition at least would be 680, so the option **A** is impossible. Consequently, they will choose **B** as the best possible answer. - It might be possible that students do not see the problem as an estimation problem, so they will use an algorithm to add. Hence, to add 28… and 4… … they will add from the right side, namely blank + blank, then 8 + blank, and finally 2 + 4 = 6 (or 7). If this last addition is 6, then the result of addition is 68…. (There is no choice). So, students will choose **B** as the answer. - Students who did not see this problem as an estimation problem might solve by an exact calculation strategy with algorithm.	- Exact (trial and error) calculation strategy ============================ Summary: No one used an estimation strategy.	37/37	100

10	9[redacted]2 489 -- A. 4[redacted] B. 56[redacted] C. 6[redacted] Students are expected to use one of the following possible strategies: - To subtract 9…2 by 489, students might look 9…2 as 900 (rounding to the nearest hundred), and 489 as 500. Hence, 900 – 500 = 400. But, they might also round off 9…2 to 1000 and 489 to 500, so 1000 – 500 = 500. Therefore, there are two possible options, **A** or **B**. The option **B** is impossible because 489 + 56… > 1000, hence the best possible answer is **A**. - Students might use front-end strategy by looking 9…2 as 9 (as 900) and 489 as 4 (as 400), then subtract 9 – 4 = 5. Hence, the option **C** is impossible. Students who stop till this thinking might choose **B** as the best possible answer. But for students who see that 56…, then add 56… + 489 which is more than 1000, will choose that the option A is the best possible answer. - Students who do not see the problem as an estimation problem might solve by an algorithm for subtraction (addition). Namely, they will subtract from the right side: 2 – 9, next blank – 8 and finally 9 – 4. Again using similar arguments as the possible strategies above, then the students will find that the option A is the best possible answer.	- Exact (trial and error) calculation strategy ============================== Summary: No one used an estimation strategy.	37/37	100
11	79 3[redacted] x [redacted] [redacted]7 + A. 2[redacted]8[redacted] B. 226[redacted] C. 33[redacted]3	- Exact trial and error calculation strategy ============================= Summary: No one used an estimation strategy.	37/37	100

	We expected that students will see 79 x 3● as an estimation problem, where it can be estimated as a multiplication 80 x 30 or 80 x 40. So, the multiplication 79 x 3● has values between around 80 x 30 = 2400 and around 80 x 40 = 3200. So, the option B and C are impossible (because 79 x 30 = 2370). Therefore, the most possible answer is the option A. Students who did not see the problem as an estimation problem might solve the problem using an exact calculation strategy. For the multiplication above, the students will make trial and error to substitute the inkblot. For example, they will do a multiplication 79 x 30, 79 x 31, 79 x 32, … or 79 x 39. Then, they might choose a possible right answer from the options based on their calculation.			
12	Noticing to the numbers of supporters and buses, then students are expected to round them to be 10.000 and 20 respectively. Therefore, each bus will bring around 500 passengers (supporters). However, we expected the students to use their knowledge or experience that the maximum total passengers in a bus is between 40 and 60 passengers. Therefore, they can conclude that the news does not make sense. Students who did not see the problem as an estimation problem might solve by an exact calculation strategy: 9998 **:** 19 = 526 or 527. Using their knowledge or experience, they can conclude that the news does not make sense. Students, who did not realize about the maximum number of passengers in a bus, only doing a computation for example, might conclude that the news makes sense.	**-** Exact calculation strategy ============================ Summary: No one used an estimation strategy.	37/37	100

Note: n = number of students used a kind of strategies; N = number of all students

Appendix 2: Several figures which were used in estimation problems



PT. MATAHARI PUTRA PRIMA . Tbk
HYPERMART BANDUNG INDAH PLAZA
NPWP/ NPPKP 21.394.0[illegible]3.5-091.000
Tgl. PKP : 21 Juli 2002

PENJUALAN 0786 000[illegible] 8635 17/05/08
286740 1[illegible]:37:30

[illegible]	2 @ KANGKUNG	3. 750
[illegible]	3 @ BAYAM	4. 550
[illegible]	1 KG KOL PUTIH	1. 675
[illegible]	1,5 KG AYAM SAYAP	14. 000
[illegible]	2 @ INDOMILK	24. 800
[illegible]	400 GRAM BAWANG MERAH	2. 990
[illegible]	250 GRAM CABE MERAH	4. 499
[illegible]	1 BTL COOKING OIL	6. 295

12 ITEMS TOTAL
TUNAI
KEMBALI

Barang Kena Pajak, Harga termasuk PPN
TERIMA KASIH
ATAS KUNJUNGAN ANDA

Figure 8.1: Receipt for Problems 1.a, 1.b, or 1

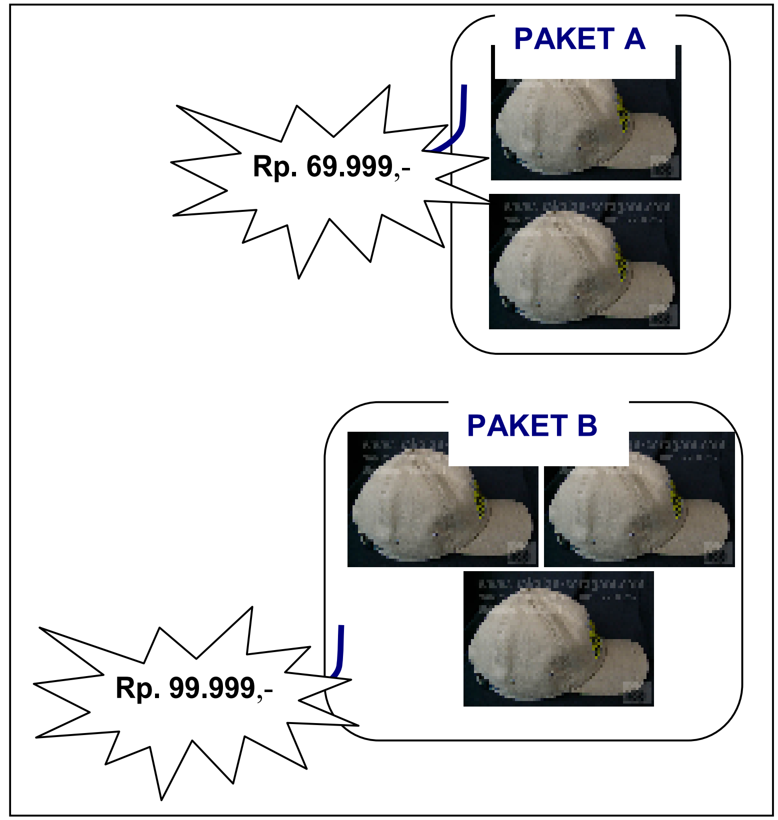


Figure 8.2: Picture for Problems 11, 6

Summary

1 Introduction and research questions

One calculation form that is used most in our daily life is computational estimation. For instance, when we are in the supermarket, we often use estimation to know how much money will be spent before going to the supermarket's cashier, and after leaving the supermarket we frequently check whether the calculation in the receipt is reasonable or not. This is an example that mathematics is actually part of our life. That is why Freudenthal (1991) said that mathematics should be seen as a human activity.

Many mathematics educators said that estimation is a very important basic skill that should be mastered by students (Reys, Rybolt, Bestgen, & Wyatt, 1982; Rubenstein, 1985) because this is useful either for solving mathematical problems at school or daily life. Moreover, according to Van den Heuvel-Panhuizen (2001), estimation has a didactical function for learning, for instance, doing estimation beforehand can help to master mental calculation strategies in arithmetic.

However, at schools, estimation has only a small place in mathematics curriculum even over the world (Reys, Bestgen, Rybolt, & Wyatt, 1982; Reys, Reys, & Penafiel, 1991). In addition, most of students seem uncomfortable with estimation (Trafton, 1986). When students are given estimation problems, they frequently solve the problems by an exact calculation.

Based on the above issues we conducted research on computational estimation with the aims: (1) *to investigate students' strategies in solving estimation problems*; and (2) *to gain insight into how students can be stimulated to use estimation strategies instead of using exact calculation in solving estimation problems.* In the light of these aims, we conducted design research with the research questions: (1) *What strategies do students use to solve estimation problems?* (2) *What are students' difficulties in solving estimation problems?* (3) *What kind of problems invite students to use estimation?* and (4) *What kind of learning-teaching situations invite students to use estimation?*

2 Theoretical framework

Computational estimation is the process of simplifying an arithmetic problem to find a satisfactory answer, without actually calculating it exactly. Regarding the learning-teaching of computational estimation, Van den Heuvel-Panhuizen (2001) distinguished three types of questions that are the driving force behind learning to estimate, namely: *(1) Are there enough? (2) Could this be correct?* and *(3) Approximately how much is it?* With regard to completeness of data from the problems, there are two kinds of estimation problems: estimation problems with complete and incomplete or unavailable data. In general, there are three cognitive processes that can be used to solve estimation problems, namely *reformulation, translation, and compensation* (Reys et al., 1982; Reys et al., 1991). Each cognitive process includes different estimation strategies. *Reformulation* includes, for instance, *rounding*, *front-end*, and *substitution* strategies. *Translation* includes, for instance, *changing operations*, and *making equivalents* strategies. And *compensation* includes *intermediate* and *final compensation.*

In our research in grade 4 and 5 we focused on: (1) an investigation of strategies used by students to solve estimation problems; (2) an understanding of students' difficulties in solving estimation problems; (3) looking for problems that invite students to use estimation; and (4) in particular for grade 5, the research is also focused on a creation of learning-teaching situations to encourage students in the use of estimation. To do these we use the theory of realistic mathematics education (RME) because it offers pedagogical and didactical both on mathematical learning and instructional materials (Treffers, 1987; Gravemeijer, 1994; Bakker, 2004).

RME is a theory of mathematics education which has been developed in the Netherlands since the 1970s and it has been extended there and also in other countries (De Lange, 1996). RME is shaped by Freudenthal's view on mathematics (Freudenthal, 1991), namely: mathematics should always be meaningful to students and should be seen as a human activity. There are five tenets of RME according to Treffers (1987) and Bakker (2004). Besides that, there are also principles which are offered by RME to design learning-teaching environments such as: *guided reinvention*, and *didactical phenomenology* (Gravemeijer, 1994). Based on the tenets and principles of RME, we designed research instruments and a learning-teaching environment for learning estimation.

3 Research methodology

In this research, design of research was a crucial part of the research. Therefore we used *design research* as the research methodology. The core of this kind of research is formed by classroom teaching experiments (Gravemeijer, 2004). In our case, the purpose of this kind of research is to answer the research questions about students' thinking processes and to design an instructional environment that supports students in learning estimation.

Design research encompasses three phases: a preliminary design, a teaching experiment, and a retrospective analysis (Gravemeijer, 2004; Bakker, 2004). A design and research instrument that proved useful during all phases of design research is called hypothetical learning trajectory (HLT), where it includes: learning goals, learning activities, and hypothetical learning process (Bakker, 2004; Simon, 1995).

In the preliminary design, we made HLT 1. This was used in the first research period: May-June 2008, for grade four (10-11 years old) of a PMRI class. The purpose of this research period was to answer the first three research questions. In addition, the research results of this period would be used to revise HLT 1 to HLT 2 that would be used in the second research period: July-August 2008. Here students were asked to solve estimation problems individually. After each lesson, at least three students were interviewed based on their answers on the worksheets.

From the first research period we found: (1) estimation strategies that used by students consists only of *rounding* and *front-end* strategies to solve computational estimation problems; and (2) students found difficulties in the use of estimation to solve: problems which numbers involved are 'too' far from the nearest tens, hundreds, thousands, or other easy numbers; inkblot multiplication problems (a kind of incomplete data problem); unavailable data problems; problems which numbers involved are decimals or fractions; and problems that need more than one step solution.

In the teaching experiment, HLT 2 was used for primary school students of the first semester of grade five (10-11 years old) of a non-PMRI class. The purposes of this research period were to get better answer to the first three research questions than in the first research period and to answer the fourth research question. Here, the students would be asked to solve estimation problems under the teacher guidance in learning-teaching situations. During the teaching experiment we would use a video camera and the researcher would also be available in the class to taking notes, pictures, and help the teacher. In addition, after each lesson, at least three students would be interviewed to know their thinking processes based on their answers on the worksheets. Therefore, we would get video, audio interview, students' worksheets, and field notes data.

In the retrospective analysis, all data during the research would be analyzed to answer the research questions. Here the HLT was compared to students' actual learning.

4 First hypothetical learning trajectory and the retrospective analysis

In this part we describe the first hypothetical learning trajectory (HLT 1) which was used during the first research period and the analysis of the results of this research period. We first describe HLT 1 that was used for primary school students of the second semester of grade four—10–11 years old. Second, we analyze the results of this research period. And third, we describe the revision of HLT 1 to HLT 2 which would be used in the second research period.

First hypothetical learning trajectory (HLT 1)

In general, we expected that students would increasingly use estimation strategies. This means during the lessons we predicted that there would be students who solve estimation problems by estimation strategies and there would also be other students who solve problems by an exact calculation strategy. We expected number of the latter kind of students would decrease from lesson to lesson. Briefly, HLT 1 is described in Table S.1 below.

Table S.1**:** An overview of HLT 1 (used in the first research period: May-June 2008)

Problems	**Type of numbers**	**Operations**	**Expected Difficulty**
1 – 5 (**a**/**b** versions)	Integers Examples: 50,000; 10,000; 5; etc.	Addition, subtraction, multiplication	**Easy**
6 – 9 (**a**/**b** versions)	Integers, decimals, simple fractions Examples: 1,675; 1.5; 1/2; etc.	Addition, subtraction, multiplication, division, and a combination of these	
10 – 15	Larger Integers, decimals, fractions Examples: 69,999; 9,998; 3/4; etc.	Addition, subtraction, multiplication, division, and a combination of these (also operations with fractions and decimals)	**Difficult**

Retrospective analysis

From students' answers, as expected in HLT 1, we found two kinds of strategies used by students to solve estimation problems, i.e., estimation strategies and exact calculation strategy. Students' answers that used only words (without mathematical reasons) or no answers at all are classified as unclear reasons. Estimation strategies which were used by students can be classified as *rounding* and *front-end* strategy. This means, in this case, the cognitive processes used by students belong to *reformulation*, but none of students used other cognitive processes: *translation* or *compensation*.

Next we present an overview of students' global performances in the use of estimation during the first research period (see Figure S.1). In general, we see that for problems 1 to 7 there is an upward trend in the use of estimation strategies, both for **a** and **b** versions. This trend, however, does not continue.

Further, we can also make the following observations: (1) there is only a small difference in the use of estimation between the **a** and **b** versions of problems 1 to 9, except for 4; and (2) there is a sudden drop in the use of estimation strategies after Problem 7. This was a contradiction to our expectation in HLT 1. We therefore go on to analyze the data in search of possible explanations.

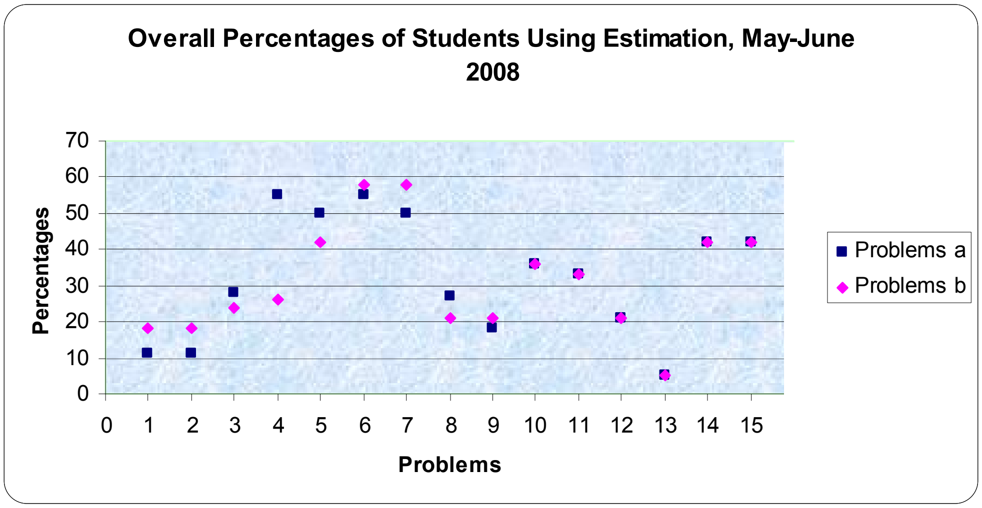


Figure S.1: Overall percentages of students using estimation in the period May-June 2008

Note: Problems 1 to 9 have **a** and **b** versions, whereas Problems 10 to 15 do not have versions.

In Problems 4 (a and b) we find out possible reasons why there is large difference between these two versions. For the second observation we analyze Problems 8, 9, 12, and 13. From the analysis we found students' difficulties in solving these problems, we also find out possible reasons why the difficulties happened to students. In addition, we can find out characteristic of problems that less invite students to use estimation. Therefore, based on this analysis, we can answer the second and the third research questions.

Revision of the HLT

Based on the retrospective analysis, we then revised the HLT 1 to HLT 2. In the revision, we reduced problems, re-ordered problems, and we decided to use the **a** versions only. The re-arrangement of problems from the first to second research period can be seen in Table S.2.

Table S.2: Order of problems used in the second research period: July-August 2008

P. May-June	1.a	8.a	3.a	4.a	10	11	14	15	6.a	7.a	12	13
P. July-August	1	2	3	4	5	6	7	8	9	10	11	12

Note: This table means, for example, Problem 1 (in the second research period) = Problem 1.a (in the first research period); Problem 2 = Problem 8.a; and so forth.

5 Second hypothetical learning trajectory and the retrospective analysis

In a similar manner to the previous part, we describe: HLT 2 that was used in the second research period, an analysis of the results of the second research period, and a proposal to revise the HLT 2 based on the analysis. Our analysis, in particular, is focused on answering four research questions.

Second hypothetical learning trajectory (HLT 2)

In general, like in the HLT 1, we expected that students would increasingly use estimation strategies from lesson to lesson. This means during the lessons we predicted that there would be students who solve estimation problems by estimation strategies and there would also be other students who solve problems by an exact calculation strategy. We expected that number of the latter kind of students would decrease from lesson to lesson except perhaps for new type of problems. Briefly, HLT 2 is described in Table S.3 below.

Table S.3: An overview of HLT 2 (used in the second period: July-August 2008)

Problems	Type of problems	Type of numbers	Operations	Expected difficulty
1 – 4	Problems with complete data	Integers Examples: 50,000; 1,675, etc.	Addition, Multiplication, Combination: division and multiplication	**Easy**
5 – 8	Problems with complete data	Larger integers, decimals and fractions Examples: 69,999; 3/4; etc.	Combination: addition, multiplication, division of integers; Combination: addition, multiplication with a (simple) fraction; Combination: addition and multiplication with fractions	
9– 12	Problems with incomplete or unavailable data	Integers Examples: 9998; 28●; 4●●, etc.	Addition, subtraction, multiplication, Combination: multiplication, division	**Difficult**

Retrospective analysis

The analysis is specifically focused on answering the four research questions. Similar to the results of the first research period, based on students' answers, students also used two kinds of strategies to solve estimation problems, namely: estimation strategies and exact calculation strategy. Estimation strategies which were also used by students were *rounding* and *front-end* strategy. This means, in this case, the cognitive processes used by students belong to *reformulation*. However, none of students used *translation* or *compensation*. From analysis of video recordings and field notes we found that the teacher did not guide the students to use other cognitive processes (*translation or compensation*) rather than *reformulation* to solve estimation problems. The difference between results in the first and second research period is: in the second research period, none of the students used any estimation strategy to solve estimation problems with incomplete or unavailable data.

Similar to the analysis of the first research period we first present an overview of overall percentages of students using estimation during the second research period (see Figure S.2).

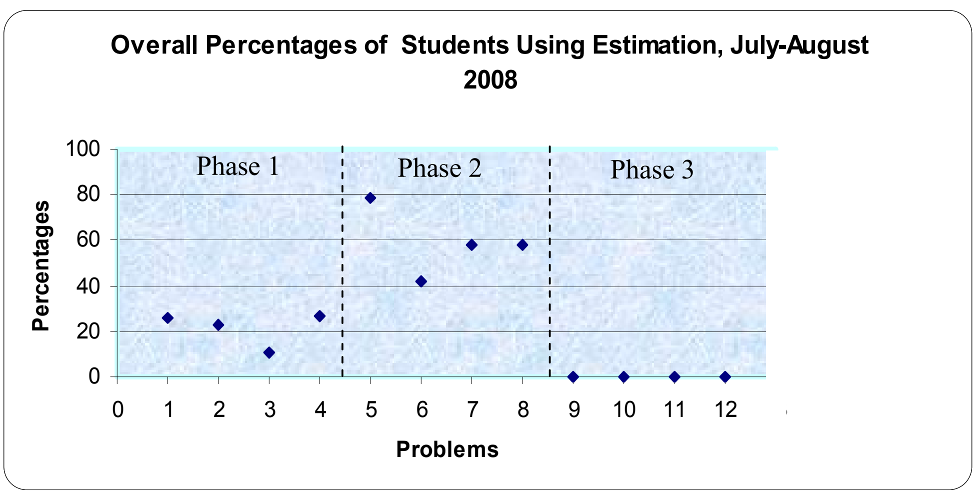


Figure S.2: Overall percentages of students using estimation in the period July-August 2008

From the graph (in Figure S.2), for problems 1 to 8 except for Problem 5, we might see an upward trend in the use of estimation strategies as we expected in

HLT 2. However, surprisingly after Problem 8 none of the students used an estimation strategy. In the HLT 2 we classified problems into three groups: Problems 1 – 4, Problems 5 – 8, and Problems 9 – 12. We in retrospect understand this classification. From the graph, we can distinguish the graph into three phases (see Figure S.2). In phase 1, the average percentages are around 21 %, which means that around 21% students solved problems using estimation strategies. In phase 2, the average percentages are around 59%, which means that around 59% students solved problems using estimation strategies. However in phase 3, it is surprising because none of the students used an estimation strategy—this result is absolutely different from the result of the first research period. Therefore, there are at least two observations that need more explanation: (1) Problem 5 evoked a sudden high percentage in the use of estimation; and (2) none of the students solved the inkblot problems (Problems 9 – 11) and an unavailable data problem (Problem 12) by any estimation strategy.

In Problems 5 we find out possible reasons why students are invited to use more estimation strategies, i.e., context of this problem(namely ice cream) is experientially real for students; the problem is open such that elicit students to use both different strategies and different answers; the problem can be solved flexibly by different operations (addition, multiplication, or division); numbers involved in the problem are easy to round off; the teacher's role and classroom situations are important in guiding students to use estimation.

For the second observation we analyzed Problems 8 and 9 because there are two large different results between the first and second research period. In the first research period these two problems—previously problems 6.a and 7.a respectively—were relatively successfully done using estimation strategies, namely 55% and 50% students solved the problems by estimation strategies respectively, however none of students solved these problems using estimation in the second period. The reasons can be the following: (1) students did not recognize the problems as estimation problems; (2) students might not think to use positional system of numbers (considering magnitudes of numbers) to solve these problems; and (3) the teacher's role is very influential to students, namely the teacher told to students how to solve the problems by exact (trial and error) calculation which are then followed by the students.

Proposal for revision of HLT 2

Based on the retrospective analysis, we revised the HLT 2 in order to invite more students to use estimation. In revising, we took into account the following factors: the mathematical problems themselves; design of problems; and classroom cultures. Mathematical problems include, for instance, operation of numbers, and type of numbers. Problems with addition and subtraction were given first, problems with multiplication and division afterward. Problems with complete data are given first, next problems with incomplete or unavailable data.

Design of problems includes, for instance, difficulties, context of problems, selection of numbers, and types of questions. Problems with more than one step solution are generally more difficult than problems with one step solution. Numbers that are close to nearest tens, hundreds, thousands, or other easy numbers can guide students to do rounding off for estimation. And questions which do not require exact answers can invite student to use estimation strategies.

Classroom cultures include, for instance, students' own strategies, teacher guidance, group and classroom discussion (interactivity in learning-teaching situations). In the learning estimation, students should be encouraged to come up with their own strategies under the teacher guidance in the interactive learning-teaching situations.

6 Conclusion and discussion

In this part we present answers to the research questions. Next we discuss research findings and give suggestions for classroom practices and future research.

Answer to the first research question

Strategies used by students to solve estimation problems from both research periods, as predicted in the HLT, can be classified into two: estimation strategies and exact calculation strategy. Estimation strategies that were used can be identified as *rounding* and *front-end* strategies, where the *rounding* strategy is used most. According to Reys et al. (1991) these estimation strategies belong to a cognitive process which is called *reformulation*. Other cognitive processes which did not emerge from students, during the research, are *translation* and *compensation.*

Answer to the second research question

We categorize students' difficulties into three, namely difficulties that were related to: (1) numbers, (2) degree of complexity of problems, and (3) students' habits. Difficulties related to numbers include: difficulties in using numbers that are not clearly factors of other numbers; rounding off numbers that are "too" far from the nearest tens, hundreds, thousands, or other easy numbers; and solving estimation problems that have to do with fractions or decimal numbers. Difficulties related to degree of complexity of problems include: number of steps to solve problems (problem with more than one step solution are generally more difficult than problems with one step solution); translating problems with a lot of information and inventing realistic data to solve problems (because students are used to solve problems with all information is given); connecting various information from the problems themselves and the outside (knowledge, experience, etc) to find an answer; thinking reflectively with what students have done (because students are used to solve problems without looking back to the answers whether reasonable or not); and solving problems that they had never encountered before. And difficulties related to students' habits include, for instance, most of students are not easy to convince to use estimation strategies in solving computational estimation problems because they need assurance that exact calculation is not necessary.

Answer to the third research question

Based on our experience with the students involved in the research period**,** we speculate that the following characteristics of computational estimation problems can invite more students to use estimation strategies: (1) contexts used in problems should be experientially real for students; (2) the problems should be open; (3) the numbers involved in the problems should initially be close enough to the nearest tens, hundreds, thousands, or other easy numbers; (4) problems can be solved using different operations (for instance addition, multiplication, etc); and (5) the questions used in the problems should not require exact answers.

Answer to the fourth research question

Based on our experience with students involved in the second research period, we found two aspects that might invite more students to use estimation in solving computational estimation problems, namely the role of the teacher and the lesson structure in the classroom learning-teaching situations. Regarding the teachers' role, we found in our case (Indonesian culture) that the students would generally follow what have been explained by the teachers. Therefore, a 'stronger' teacher, in guiding to influence students to use estimation strategies, is important. Regarding the lesson structure of the classroom learning-teaching situations, in particular, we could say that group and class discussions might motivate students to share opinion, justification, and (estimation) strategies.

Discussion

In this part we discuss: students' estimation strategies, inkblot problems, bus problem, preparing teacher(s) for conducting design research in Indonesian cultures, teachers' role and classroom cultures' differences between PMRI and RME, and possible future research.

- *Students' estimation strategies.* As we mentioned earlier, the estimation strategies used by students only include *rounding* and *front-end* strategy. One reason why only these strategies emerged in students' answers is because the problems which were used do not clearly invite students to use other strategies. Therefore, we need to improve or use other estimation problems. One possible way in improving problems is by changing the nature of numbers involved in the problems—so the problems invite students to use other estimation strategies (e.g. 3750 + 1675 + 2990 + 4990 to be 3950 + 3975 + 4090 + 4019. Therefore, we can solve this as 4 x 4000, for instance, besides it can also be solved by 4000 + 4000 + 4000 + 4000 =…)

- *Inkblot problems.* One possible reason why students find it difficult to solve inkblot problems is because the students might not think of positional system of numbers (students are used to work addition, subtraction, or multiplication from the right to the left, whereas an estimation makes more sense to work from the left to the right, for instance, from hundreds to tens, and to units). We suggest three ways to improve the inkblot problems in order to make the students would be aware of positional system of numbers. First we might simplify the problems by reducing the inkblots. Second, we could change the format of problems without reducing the inkblots, namely from a column addition to a row addition, from column multiplication to a row multiplication, etc. And third, we can combine the first and second ways.

  We found interesting students' answers that used a rather different strategy, namely excluding impossible options to find an answer. In the analysis this strategy is classified as an exact (trial and error) calculation strategy. However, we think this strategy is different from the common exact calculation because when we are excluding impossible options, actually we are using different cognitive processes rather than cognitive processes that happen if we use the exact calculation strategy. After all, we do not know yet what kind of cognitive processes which is used in this strategy
- *Bus problem.* We found in the first research period the students were less tempted to solve the bus problem by estimation strategies than by exact calculation strategy. More surprisingly, none of students solved the bus problem by estimation strategies in the second research period. One possible reason is because most of the students have not been trained to think reflectively: they are only used to doing calculation, without looking back to check the calculation results. To make students think reflectively we should slightly change the question. Do not ask **whether make sense or not the news** but we change it to be **could the number of buses bring 9998 supporters?**

  Another possible reason is because students are not used to combine information from the problem itself and from outside the problem—in this case real-world knowledge or experience. This might be caused by students' view that mathematics (arithmetic) and real-world contexts are separate systems. Thus, in learning-teaching situations, we think teachers should give opportunity to students to solve problems that combine information from the problems themselves and outside the problems. Moreover, giving rich context problems to students hopefully would change their view: from the view that mathematics and contexts are separate systems to a new view that mathematics and context can be connected.

In the PMRI class, students might have used to solve contextual problems which combine information from problems themselves and outside the problems—this might explain why there were students solved the bus problem by estimation strategies. In the non-PMRI class, however, students are not used to solve problems which combine information from the problems themselves and outside the problems—this might also explain why none of students use estimation strategies to solve the bus problem.

- *Preparing teacher(s) for conducting design research in Indonesian cultures.*
  One of the three phases in design research is the teaching experiment. We tried to implement the teaching experiment based on the plan. However, based on field notes and video recordings, we found difficulties particularly concerning preparation of the teacher as indicated in the following: (1) The teacher did not always follow the plan in teaching-learning situations, for example, in introducing lessons, the teacher sometimes used different contextual situations from the context that used in the problems; (2) We assumed that the teacher understood the philosophy of RME. However, in our view, the teacher gave too much guidance to students for instance she told to students how to solve the inkblot problems. This implies students did not solve problems on their own: they followed the teacher's strategy.

  Since in Indonesian cultures we should give a great respect to the teacher, we then were reluctant to give suggestions—it is impolite. Moreover, since we are younger than the teacher, we should very appreciate to the teacher's decisions. Such cultural issues are important to take into account when we try to implement design research in educational practice, particularly in the teaching experiment phase in Indonesia. This also could be a consideration when we will conduct co-design research in the PMRI project, for example.

- *Teachers' role and classroom cultures' differences between PMRI and RME.* RME is developed in the Netherlands and PMRI is the Indonesian version of RME. Although between these two countries had a very close connection in the past, they have very different cultures. This might happen also in educational practice, such as follows: (1) Classroom social norms that generally established in Indonesian situations are: in general students are not used to expressing their thinking in front of the class, students are reluctant to ask questions to the teacher if they do not understand yet, students try to avoid different arguments either with the teacher or other students that expressed directly in the class. This could be in contrast if we compare to the Dutch students; and (2) another classroom social norm that established particularly in the non-PMRI class is that students are dependent on the teacher's explanation: students will follow what the teacher told to students. This might be different from the Dutch situations: Dutch students might not follow everything from the teacher's explanation and they are more critical and perhaps less polite, where they seen more used to think for themselves.

  Therefore, such potential differences are important to take into account when we try to implement a RME approach in Indonesian classroom situations through PMRI.
- *Possible future research.* In this research we used paper and pencil to find out how students solve computational estimation problems. However, in daily life, we frequently use estimation strategies mentally to solve problems that are encountered. Thus, we propose that the research on estimation might be enhanced with estimation problems without paper and pencils to make them more experientially real to students and avoid the possibility of exact calculations. Whether this will work in Indonesian context is an interesting topic for future research.

Abstract

One calculation form that is used most in our daily life is computational estimation. This basic skill is suggested by many mathematics educators to be mastered by students (Reys, Rybolt, Bestgen, & Wyatt, 1982; Rubenstein, 1985). However, it has only a small place in mathematics curriculum over the world (Reys, Bestgen, Rybolt, & Wyatt, 1982; Reys, Reys, & Penafiel, 1991), particularly in Indonesia. Based on this issue we conducted a research with the aims: (1) to investigate students' strategies in solving estimation problems; and (2) to gain insight into how students can be stimulated to use estimation strategies instead of using exact calculation in solving estimation problems. Because design is a crucial part in this research, we used design research as the research methodology in the frame of realistic mathematics education. The research was conducted in two periods: The first period was for primary Indonesian students of grade four (10-11 years old), and the second period was for primary Indonesain students of grade five (10-11 years old). From both research periods we found: (1) estimation strategies that are used to solve computational estimation problems consist of only *rounding* and *front-end* strategy; (2) students' difficulties in learning computational estimation can be related to number relationships, complexity of problems, and students' habits; (3) characteristics of problems that invited students to use more estimation strategies are: contexts should be experientially real for students; problems are open; numbers involved in problems are close to the nearest tens, hundreds, thousands, or other easy numbers; and questions which are used in problems should not require exact answers; and (4) two aspects that should be considered in learning-teaching estimation are the role of the teacher and the lesson structure in learning-teaching situations. These findings are discussed either for use in learning-teaching situations or for future research.

Keywords: computational estimation, estimation strategies, design research, realistic mathematics education.